

# A CATEGORICAL APPROACH TO GROUPOID FROBENIUS ALGEBRAS

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ABSTRACT. In this paper, we show that  $\mathcal{G}$ -Frobenius algebras (for  $\mathcal{G}$  a finite groupoid) correspond to a particular class of Frobenius objects in the representation category of  $D(k[\mathcal{G}])$ , where  $D(k[\mathcal{G}])$  is the Drinfeld double of the quantum groupoid  $k[\mathcal{G}]$  [9].

## 1. INTRODUCTION

Groupoid Frobenius algebras were introduced recently in [12] as a groupoid version of (non-projective)  $G$ -Frobenius algebras ( $G$ -FAs) for  $G$  a finite group [13] [7]. As shown<sup>1</sup> in [13],  $G$ -FAs are the algebraic structures which classify certain homotopy quantum field theories (HQFTs). Roughly speaking, a  $(d+1)$ -dimensional HQFT is a topological quantum field theory [1] for  $d$ -dimensional manifolds and  $(d+1)$ -dimensional cobordisms endowed with homotopy classes of maps into a given space  $X$ . In the case when  $X$  is an Eilenberg-MacLane space of type  $K(G, 1)$ , one finds that the associated  $(1+1)$ -dimensional HQFTs are classified by  $G$ -FAs [13].

The author's original motivation for generalizing  $G$ -FAs to  $\mathcal{G}$ -FAs for  $\mathcal{G}$  a finite groupoid was the existence of certain "atypical"  $G$ -FAs<sup>2</sup> that were originally constructed in [8]. In addition to satisfying the usual  $G$ -FA axioms, these "atypical"  $G$ -FAs were also endowed with certain other features which were not characteristic of  $G$ -FAs in general. These examples motivate the view that  $G$ -FAs are actually a special case of some larger algebraic structure.

Ultimately, it was the transition from group to groupoid that resulted in a framework that was capable of accommodating these examples. Within the  $\mathcal{G}$ -FA framework, one can show that the aforementioned  $G$ -FAs are actually derived from a certain kind of (non-trivial) groupoid Frobenius algebra. As it turned out, these early motivating examples were just the tip of the iceberg. It was shown in [12] that by working within the  $\mathcal{G}$ -FA framework, one could construct a tower of increasingly complex  $G$ -FAs, where each  $G$ -FA in the tower is derived from some groupoid Frobenius algebra. In addition to this,  $\mathcal{G}$ -FAs could also be used to gain new insight on the problem of twisting ordinary  $G$ -FAs.

<sup>1</sup>For an alternate approach to  $G$ -FAs, see [7].

<sup>2</sup>The  $G$ -FAs in question are those arising from the stringy cohomology or stringy  $K$ -theory [5] of the inertia manifold. See section 4 of [8] for additional details.

It was shown in [6] that every  $G$ -FA has a twist by any element of  $Z^2(G, k^\times)$ . Since these twists apply to all  $G$ -FAs, one can regard them as “universal”  $G$ -FA twists. In an analogous manner,  $\mathcal{G}$ -FAs have their own universal twists where the twisting is now by the elements of  $Z^2(\mathcal{G}, k^\times)$  [12]. When one combines this point with the aforementioned tower of “ $\mathcal{G}$ -FA induced”  $G$ -FAs, one obtains a significant generalization of the  $G$ -FA twisting result from [6]. Specifically, for every  $n \geq 2$ , one can always find a class of  $G$ -FAs with twists by any element in  $Z^n(G, k^\times)$  [12].

While [12] illustrates the utility of  $\mathcal{G}$ -FAs in addressing and solving these problems, little was done in [12] to motivate the choice of axioms for a  $\mathcal{G}$ -FA. The only motivation for the axioms came in the form of a short remark<sup>3</sup> which asserted that  $\mathcal{G}$ -FAs might actually correspond to certain kinds of Frobenius objects in  $\text{Rep}(D(k[\mathcal{G}]))$  (the representation category of  $D(k[\mathcal{G}]))$ , where  $D(k[\mathcal{G}])$  is the Drinfeld double of the quantum groupoid (weak Hopf algebra)  $k[\mathcal{G}]$  [9]. This assertion is motivated by a recent categorical result for  $G$ -FAs [8] which showed that  $G$ -FAs correspond to certain kinds of Frobenius objects in  $\text{Rep}(D(k[G]))$ , where  $D(k[G])$  is the original Drinfeld double of the Hopf algebra  $k[G]$  [4]. Consequently, if the assertion proves true, the  $\mathcal{G}$ -FA axioms of [12] would essentially be a consequence of generalizing  $D(k[G])$  to  $D(k[\mathcal{G}])$ . In other words, from this categorical vantage point, the notion of a  $\mathcal{G}$ -FA is a natural generalization of a  $G$ -FA for the case when  $G$  is replaced by  $\mathcal{G}$ .

With the current paper, we show that the assertion of [12] is indeed true.

The rest of the paper is organized as follows. In section 2, we give a brief review of quantum groupoids [2] [3] [11] and their representation category [9]. Then, in section 3, we prove the assertion raised in [12].

## 2. PRELIMINARIES

Throughout this paper, we use the following notation.

$k$  is a field of characteristic 0.

$\mathcal{G} = (\mathcal{G}_0, \mathcal{G}_1)$  denotes a finite groupoid whose set of objects is  $\mathcal{G}_0$  and whose set of morphisms is  $\mathcal{G}_1$ .

The source and target maps from  $\mathcal{G}_1$  to  $\mathcal{G}_0$  are denoted as  $s$  and  $t$  respectively.

For  $x \in \mathcal{G}_0$ ,  $e_x$  denotes the identity morphism associated to  $x$ .

For  $x \in \mathcal{G}_0$ ,  $\Gamma^x$  is the group consisting of all  $g \in \mathcal{G}_1$  with  $s(g) = t(g) = x$ .

**2.1. Quantum Groupoids.** Quantum groupoids or weak Hopf algebras [2] [3] [11] generalize the notion of ordinary Hopf algebras by weakening the axioms concerning the coproduct and counit in the following way:

1. the coproduct is not necessarily unit-preserving;

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<sup>3</sup>More specifically, Remark 3.1 of [12].

2. the counit is not necessarily multiplicative.

Formally, a quantum groupoid is defined as follows:

**Definition 2.1.** A quantum groupoid over a field  $k$  is a tuple  $(H, \cdot, 1, \Delta, \varepsilon, S)$  where

- (i)  $H$  is a finite dimensional unital associative algebra over  $k$  with product  $\cdot$  and unit  $1$ .
- (ii)  $H$  is a finite dimensional counital coassociative algebra over  $k$  with coproduct  $\Delta : H \rightarrow H \otimes_k H$  and counit  $\varepsilon : H \rightarrow k$ .
- (iii) The algebra and coalgebra structure of  $H$  satisfy the following compatibility conditions.
  - (a) Multiplicativity of the coproduct: for all  $x, y \in H$ ,

$$\Delta(x \cdot y) = \Delta(x) \cdot \Delta(y)$$

- (b) Weak multiplicativity of the counit: for all  $x, y, z \in H$ ,

$$\varepsilon(x \cdot y \cdot z) = \varepsilon(x \cdot y_{(1)})\varepsilon(y_{(2)} \cdot z)$$

$$\varepsilon(x \cdot y \cdot z) = \varepsilon(x \cdot y_{(2)})\varepsilon(y_{(1)} \cdot z)$$

- (c) Weak comultiplicativity of the unit:

$$(\Delta \otimes id_H) \circ \Delta(1) = (\Delta(1) \otimes 1) \cdot (1 \otimes \Delta(1))$$

$$(\Delta \otimes id_H) \circ \Delta(1) = (1 \otimes \Delta(1)) \cdot (\Delta(1) \otimes 1)$$

- (iv)  $S : H \rightarrow H$  is a  $k$ -linear map called the antipode which satisfies the following for all  $x \in H$ :
  - (a)  $x_{(1)} \cdot S(x_{(2)}) = \varepsilon(1_{(1)} \cdot x)1_{(2)}$
  - (b)  $S(x_{(1)}) \cdot x_{(2)} = 1_{(1)}\varepsilon(x \cdot 1_{(2)})$
  - (c)  $S(x_{(1)}) \cdot x_{(2)} \cdot S(x_{(3)}) = S(x)$

**Remark 2.2.** In Definition 2.1, Sweedler notation was applied so that  $\Delta(a)$  is written as  $\Delta(a) = a_{(1)} \otimes a_{(2)}$ .

**Remark 2.3.** Its a straightforward exercise to show the following:

1. Every Hopf algebra is a quantum groupoid.
2. For a quantum groupoid  $H$ , the following statements are equivalent:
  - (i)  $H$  is a Hopf algebra
  - (ii)  $\Delta(1) = 1 \otimes 1$
  - (iii)  $\varepsilon(x \cdot y) = \varepsilon(x)\varepsilon(y)$  for all  $x, y \in H$

**Example 2.4.** Any finite groupoid  $\mathcal{G}$  defines a quantum groupoid

$$(k[\mathcal{G}], \cdot, 1, \Delta, \varepsilon, S)$$

where

1.  $k[\mathcal{G}] := \bigoplus_{g \in \mathcal{G}_1} kg$  as a vector space over  $k$
2.  $\cdot$  is the multiplication on  $k[\mathcal{G}]$  induced by the composition of morphisms, that is, for  $g, h \in \mathcal{G}_1$ ,  $g \cdot h = gh$  if  $s(g) = t(h)$  and  $g \cdot h = 0$  if  $s(g) \neq t(h)$

3.  $1 := \sum_{x \in \mathcal{G}_0} e_x$
4.  $\Delta : k[\mathcal{G}] \rightarrow k[\mathcal{G}] \otimes_k k[\mathcal{G}]$  is the  $k$ -linear map induced by  $g \mapsto g \otimes g$  for all  $g \in \mathcal{G}_1$
5.  $\varepsilon : k[\mathcal{G}] \rightarrow k$  is the  $k$ -linear map induced by  $g \mapsto 1_k$  for all  $g \in \mathcal{G}_1$  where  $1_k$  is the unit element of  $k$
6.  $S : k[\mathcal{G}] \rightarrow k[\mathcal{G}]$  is the  $k$ -linear map induced by  $g \mapsto g^{-1}$  for all  $g \in \mathcal{G}_1$ .

We conclude this section by recalling a few things about *quasitriangular* quantum groupoids [9]; we begin with its definition.

**Definition 2.5.** A quasitriangular quantum groupoid is a tuple  $(H, \cdot, 1, \Delta, \varepsilon, S, R)$  where

- (i)  $(H, \cdot, 1, \Delta, \varepsilon, S)$  is a quantum groupoid, and
- (ii)  $R \in \Delta^{op}(1)(H \otimes_k H)\Delta(1)$  satisfies the following conditions for all  $h \in H$ :

$$\Delta^{op}(h)R = R\Delta(h) \quad (1)$$

$$(id_H \otimes \Delta)(R) = R_{13} \cdot R_{12} \quad (2)$$

$$(\Delta \otimes id_H)(R) = R_{13} \cdot R_{23} \quad (3)$$

where  $\Delta^{op}$  is the opposite coproduct,  $R_{12} = R \otimes 1$ ,  $R_{23} = 1 \otimes R$ , and  $R_{13} = R^{(1)} \otimes 1 \otimes R^{(2)}$ . In addition, there exists  $\overline{R} \in \Delta(1)(H \otimes_k H)\Delta^{op}(1)$  such that

$$R \cdot \overline{R} = \Delta^{op}(1) \quad (4)$$

$$\overline{R} \cdot R = \Delta(1). \quad (5)$$

**Remark 2.6.** In Definition 2.5, the R-matrix  $R$  was written as

$$R = R^{(1)} \otimes R^{(2)}$$

to simplify notation.

A Drinfeld double construction was introduced in [9] for generating quasitriangular quantum groupoids from existing quantum groupoids. When this construction is applied to the quantum groupoid  $k[\mathcal{G}]$ , the result is the quasitriangular quantum groupoid  $D(k[\mathcal{G}])$  which is defined as follows:

1. As a vector space over  $k$ ,  $D(k[\mathcal{G}])$  has basis

$$\{\gamma_g^x \mid g, x \in \mathcal{G}_1, s(g) = t(g) = t(x)\}. \quad (6)$$

2. For  $\gamma_g^x, \gamma_h^y \in D(k[\mathcal{G}])$ , the multiplication law is given by

$$\gamma_g^x \cdot \gamma_h^y := \delta_{x^{-1}gx, h} \gamma_g^{xy}. \quad (7)$$

(Note that when  $x^{-1}gx = h$ ,  $xy$  is defined since  $s(x) = s(h) = t(y)$ ).

3. The unit of  $D(k[\mathcal{G}])$  is

$$1 = \sum_{x \in \mathcal{G}_0} 1^x \quad (8)$$

where

$$1^x := \sum_{g \in \Gamma^x} \gamma_g^{e_x}. \quad (9)$$

4. The coproduct of  $D(k[\mathcal{G}])$  is defined as

$$\Delta_D(\gamma_g^x) := \sum_{\{g_1, g_2 \in \Gamma^{t(x)} \mid g_1 g_2 = g\}} \gamma_{g_1}^x \otimes \gamma_{g_2}^x \quad (10)$$

5. The counit of  $D(k[\mathcal{G}])$  is defined as

$$\varepsilon_D(\gamma_g^x) = \delta_{g, xx^{-1}}. \quad (11)$$

6. The antipode of  $D(k[\mathcal{G}])$  is defined as

$$S(\gamma_g^x) = \gamma_{x^{-1}g^{-1}x}^{x^{-1}} \quad (12)$$

7. The R-matrix is

$$R := \sum_{x \in \mathcal{G}_0} R^x \quad (13)$$

where

$$R^x := \sum_{g, h \in \Gamma^x} \gamma_g^{e_x} \otimes \gamma_h^g. \quad (14)$$

**Remark 2.7.** Note that unless  $\mathcal{G}$  has a single object,  $\Delta_D$  does not preserve the unit since

$$\Delta_D(1^x) = \sum_{g \in \Gamma^x} \Delta_D(\gamma_g^{e_x}) = \sum_{g \in \Gamma^x} \sum_{\{g_1, g_2 \in \Gamma^x \mid g_1 g_2 = g\}} \gamma_{g_1}^{e_x} \otimes \gamma_{g_2}^{e_x} = 1^x \otimes 1^x \quad (15)$$

and

$$\Delta_D(1) = \sum_{x \in \mathcal{G}_0} \Delta_D(1^x) = \sum_{x \in \mathcal{G}_0} 1^x \otimes 1^x \neq \sum_{x, y \in \mathcal{G}_0} 1^x \otimes 1^y = 1 \otimes 1. \quad (16)$$

So by Remark 2.3,  $D(k[\mathcal{G}])$  is a Hopf algebra only when  $\mathcal{G}$  is a one-object groupoid (i.e., a group). In the case when  $\mathcal{G}$  is the one-object groupoid whose set of morphisms is the group  $G$ ,  $D(k[\mathcal{G}])$  is exactly  $D(k[G])$ , the ordinary Drinfeld double of the Hopf algebra  $k[G]$ .

**2.2. Quantum Groupoids & Category Theory.** It was shown in [10] that for a quantum groupoid  $H$ ,  $\text{Rep}(H)^4$  is a monoidal category. To define the monoidal product, let  $(\rho_1, A_1)$  and  $(\rho_2, A_2)$  be objects of  $\text{Rep}(H)$ . Then

$$(\rho_1, A_1) \otimes (\rho_2, A_2) := (\rho_{12}, A_1 \hat{\otimes} A_2) \quad (17)$$

where the  $H$ -action  $\rho_{12}$  is induced by the coproduct  $\Delta$  of  $H$  via

$$\rho_{12}(h) := [\rho_1 \otimes \rho_2] \circ \Delta(h), \quad h \in H \quad (18)$$

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<sup>4</sup> $\text{Rep}(H)$  is the category whose objects are finite dimensional left  $H$ -modules and whose morphisms are  $H$ -linear maps.

and

$$\begin{aligned} A_1 \widehat{\otimes} A_2 &:= \{a \in A_1 \otimes_k A_2 \mid \rho_{12}(1)a = a\} \\ &= \rho_{12}(1)(A_1 \otimes_k A_2) \\ &= [\rho_1(1_{(1)}) \otimes \rho_2(1_{(2)})](A_1 \otimes_k A_2) \end{aligned} \quad (19)$$

where the second equality follows from the fact that  $\Delta(1) \cdot \Delta(1) = \Delta(1)$ . The monoidal product of morphisms is simply the restriction of the usual tensor product of linear maps.

For the unit object, let  $\varepsilon_t : H \rightarrow H$  be defined by

$$\varepsilon_t(h) := \varepsilon(1_{(1)} \cdot h)1_{(2)} \quad (20)$$

where  $h \in H$  and  $\varepsilon$  is the counit of  $H$ . Then the unit object of  $\text{Rep}(H)$  is  $I = (\sigma_t, H_t)$  where

$$H_t := \varepsilon_t(H), \quad (21)$$

and for  $h \in H$  and  $z \in H_t$ ,

$$\sigma_t(h)z := \varepsilon_t(h \cdot z). \quad (22)$$

If  $(\rho_i, A_i)$  are objects of  $\text{Rep}(H)$  for  $i = 1, 2$ , and  $3$ , then the associator

$$\Phi_{123} : (A_1 \widehat{\otimes} A_2) \widehat{\otimes} A_3 \xrightarrow{\sim} A_1 \widehat{\otimes} (A_2 \widehat{\otimes} A_3) \quad (23)$$

is the trivial one.

To define the left\right unit morphisms, let  $(\rho, A)$  be an object of  $\text{Rep}(H)$ . Then the left morphism

$$l_A : H_t \widehat{\otimes} A \xrightarrow{\sim} A \quad (24)$$

is defined by

$$l_A(\sigma_t(1_{(1)})z \otimes \rho(1_{(2)})a) := \rho(z)a \quad (25)$$

where  $z \in H_t$  and  $a \in A$ ; the right morphism

$$r_A : A \widehat{\otimes} H_t \xrightarrow{\sim} A \quad (26)$$

is defined by

$$r_A(\rho(1_{(1)})a \otimes \sigma_t(1_{(2)})z) := \rho(S(z))a \quad (27)$$

where  $z \in H_t$ ,  $a \in A$ , and  $S$  is the antipode of  $H$ .

If  $H$  is also quasitriangular with  $R$ -matrix  $R$ , then  $\text{Rep}(H)$  is also braided [9]. For any two objects  $(\rho_1, A_1)$  and  $(\rho_2, A_2)$  of  $\text{Rep}(H)$ , the braiding

$$c_{A_1, A_2} : A_1 \widehat{\otimes} A_2 \rightarrow A_2 \widehat{\otimes} A_1 \quad (28)$$

is defined by

$$c_{A_1, A_2}(x) := \rho_2(R^{(2)})x^{(2)} \otimes \rho_1(R^{(1)})x^{(1)} \quad (29)$$

where  $x = x^{(1)} \otimes x^{(2)} \in A_1 \widehat{\otimes} A_2$ .

**2.2.1. Frobenius Objects.** Throughout this section,  $(\mathcal{C}, \otimes, I, \Phi, l, r, c)$  will denote a braided monoidal category where  $\mathcal{C}$  is a small category,  $\otimes$  is the monoidal product,  $I$  is the unit object,  $\Phi$  is the associator,  $l$  and  $r$  are the left and right identity maps, and  $c$  is the braiding.

**Definition 2.8.** An algebra object is a tuple  $(A, m, \mu)$  where

- $A$  is an object of  $\mathcal{C}$ ,
- $m : A \otimes A \rightarrow A$  is a morphism of  $\mathcal{C}$  called the product, and
- $\mu : I \rightarrow A$  is a morphism of  $\mathcal{C}$  called the unit

which satisfy the following two conditions:

1.  $m \circ (id_A \otimes m) \circ \Phi_{A,A,A} = m \circ (m \otimes id_A)$  (associativity)
2.  $m \circ (\mu \otimes id_A) = l_A$ ,  $m \circ (id_A \otimes \mu) = r_A$  (unit property)

$(A, m, \mu)$  is said to be commutative if  $m \circ c_{A,A} = m$ .

**Definition 2.9.** A coalgebra object is a tuple  $(A, \Delta, \varepsilon)$  where

- $A$  is an object of  $\mathcal{C}$ ,
- $\Delta : A \rightarrow A \otimes A$  is a morphism of  $\mathcal{C}$  called the coproduct, and
- $\varepsilon : A \rightarrow I$  is a morphism of  $\mathcal{C}$  called the counit

which satisfy the following two conditions:

1.  $(id_A \otimes \Delta) \circ \Delta = \Phi_{A,A,A} \circ (\Delta \otimes id_A) \circ \Delta$  (coassociativity)
2.  $l_A \circ (\varepsilon \otimes id_A) \circ \Delta = id_A = r_A \circ (id_A \otimes \varepsilon) \circ \Delta$  (counit property)

$(A, \Delta, \varepsilon)$  is said to be co-commutative if  $c_{A,A} \circ \Delta = \Delta$ .

**Definition 2.10.** A Frobenius object is a tuple  $(A, m, \Delta, \mu, \varepsilon)$  where  $(A, m, \mu)$  is a commutative algebra object and  $(A, \Delta, \varepsilon)$  is a co-commutative coalgebra object which satisfies

$$\Delta \circ m = (m \otimes id_A) \circ \Phi_{A,A,A}^{-1} \circ (id_A \otimes \Delta) \quad (30)$$

$$\Delta \circ m = (id_A \otimes m) \circ \Phi_{A,A,A} \circ (\Delta \otimes id_A). \quad (31)$$

**Remark 2.11.** Throughout this paper, we will disregard  $\Phi$  from our expressions since  $\text{Rep}(D(k[\mathcal{G}]))$  (our of category of interest) has a trivial associator.

### 3. $\mathcal{G}$ -FAS & FROBENIUS OBJECTS IN $\text{Rep}(D(k[\mathcal{G}]))$

We begin this section by recalling the axiomatic definition of a  $\mathcal{G}$ -FA [12]:

**Definition 3.1.** A  $\mathcal{G}$ -Frobenius Algebra ( $\mathcal{G}$ -FA) is given by the following data

$$\langle \mathcal{G}, (A, \bullet, \mathbf{1}_A), \eta, \varphi \rangle$$

where

- (a)  $\mathcal{G} = (\mathcal{G}_0, \mathcal{G}_1)$  is a finite groupoid.

- (b)  $(A, \bullet, \mathbf{1}_A)$  is a finite dimensional associative algebra over  $k$  with product  $\bullet$  and unit  $\mathbf{1}_A$  which splits as a direct sum of algebras which are indexed by the objects of  $\mathcal{G}$ :

$$A = \bigoplus_{x \in \mathcal{G}_0} A^x. \quad (32)$$

- (c)  $\eta : A \times A \rightarrow k$  is a bilinear form.  
 (d)  $\varphi$  is a  $\mathcal{G}$ -action which acts on  $A$  by algebra homomorphisms, that is, if (1)  $x \in \mathcal{G}_1$  with  $s(x) = x$  and  $t(x) = y$ , then  $\varphi(x) : A^x \rightarrow A^y$  is an algebra isomorphism, (2) if  $g, h \in \mathcal{G}_1$  and  $s(h) = t(g)$ , then  $\varphi(h) \circ \varphi(g) = \varphi(hg)$ , and (3)  $\varphi(e_x) = id_{A^x}$

which satisfies the following for all  $x, y \in \mathcal{G}_0$ :

- (i)  $A^x = \bigoplus_{g \in \Gamma^x} A_g^x$  is a  $\Gamma^x$ -graded algebra.
- (ii) if  $a^x \in A^x$  and  $b^y \in A^y$ , then  $a^x \bullet b^y = \delta_{x,y} a^x \bullet b^y \in A^x$ .
- (iii)  $\eta(a \bullet b, c) = \eta(a, b \bullet c)$  for all  $a, b, c \in A$ .
- (iv)  $\eta(\varphi(h)a^x, \varphi(h)b^x) = \eta(a^x, b^x)$  for all  $a^x, b^x \in A^x$  and  $h \in \mathcal{G}_1$  satisfying  $s(h) = x$ .
- (v)  $\varphi(x)A_g^x \subset A_{xgx^{-1}}^y$  for a morphism  $x \in \mathcal{G}_1$  satisfying  $s(x) = x$  and  $t(x) = y$ .
- (vi)  $\eta|_{A_g^x \times A_h^x}$  is nondegenerate for  $gh = e_x$  and zero otherwise.
- (vii)  $a_g^x \bullet a_h^x = (\varphi(g)a_h^x) \bullet a_g^x \in A_{gh}^x$  for  $a_g^x \in A_g^x$  and  $a_h^x \in A_h^x$ .
- (viii)  $\varphi(g)|_{A_g^x} = id_{A_g^x}$
- (ix) if  $g, h \in \Gamma^x$ ,  $c \in A_{ghg^{-1}h^{-1}}^x$ , and  $l_c : A \rightarrow A$  is the linear map induced by left multiplication by  $c$ , then

$$\text{Tr} \left( l_c \circ \varphi(h) |_{A_g^x} : A_g^x \rightarrow A_g^x \right) = \text{Tr} \left( \varphi(g^{-1}) \circ l_c |_{A_h^x} : A_h^x \rightarrow A_h^x \right) \quad (33)$$

where  $\text{Tr}$  denotes the trace.

**Remark 3.2.** The definition of groupoid Frobenius algebras given in [12] was stated in terms of group Frobenius algebras. In an effort to make Definition 3.1 self contained, we have reworded the original definition of [12] to avoid any reference to group Frobenius algebras.

**Remark 3.3.** In the special case when  $\mathcal{G}$  is the one-object groupoid whose set of morphisms is the group  $G$ , Definition 3.1 reduces to the definition of a  $G$ -Frobenius algebra.

We now state the main result of this paper:

**Theorem 3.4.** *Every  $\mathcal{G}$ -FA is derived from a Frobenius object  $((\rho, A), m, \Delta, \mu, \varepsilon)$  in  $\text{Rep}(D(k[\mathcal{G}]))$  which satisfies*

- (1)  $\sum_{x \in \mathcal{G}_0} \sum_{g \in \Gamma^x} \rho(\gamma_g^g) = id_A$
- (2)  $\text{Tr}(l_c \circ \rho(\gamma_{hgh^{-1}}^h)) = \text{Tr}(\rho(\gamma_h^{g^{-1}}) \circ l_c \circ \rho(\gamma_h^{e_x})) \forall x \in \mathcal{G}_0, g, h \in \Gamma^x$ , and  $c \in \rho(\gamma_{ghg^{-1}h^{-1}}^{e_x})A$  where  $\text{Tr}$  denotes the trace and  $l_c$  is the  $k$ -linear map defined by  $l_c(v) = m(c \otimes v)$  for  $v \in \rho(1^x)A$ .



In addition, every Frobenius object in  $\text{Rep}(D(k[\mathcal{G}]))$  which satisfies conditions (1) and (2) induces a  $\mathcal{G}$ -FA.

We now dedicate the remainder of the paper to the proof of Theorem 3.4.

**3.1.  $\mathcal{G}$ -FAs via Frobenius objects.** In this section, we show that every Frobenius object in  $\text{Rep}(D(k[\mathcal{G}]))$  which satisfies conditions (1) and (2) of Theorem 3.4 corresponds to a particular  $\mathcal{G}$ -FA.

We begin with the following result which shows that every left  $D(k[\mathcal{G}])$ -module has a direct sum decomposition which resembles that of a  $\mathcal{G}$ -FA.

**Proposition 3.5.** *Let  $(\rho, A)$  be a left  $D(k[\mathcal{G}])$ -module. Then*

- (i)  *$A$  has a direct sum decomposition  $A = \bigoplus_{x \in \mathcal{G}_0} A^x$  which is indexed by the objects of  $\mathcal{G}$  where  $A^x := \rho(1^x)A$ . In particular,  $\rho(1^y)a^x = \delta_{x,y}a^x$  for  $a^x \in A^x$ .*
- (ii) *For each  $x \in \mathcal{G}_0$ ,  $A^x$  has a direct sum decomposition  $A^x = \bigoplus_{g \in \Gamma^x} A_g^x$  where  $A_g^x := \rho(\gamma_g^{e_x})A^x = \rho(\gamma_g^{e_x})A$ . In particular,  $\rho(\gamma_h^{e_y})a_g^x = \delta_{g,h}a_g^x$  for  $a_g^x \in A_g^x$ ,  $y \in \mathcal{G}_0$ , and  $h \in \Gamma^y$ .*
- (iii) *For all  $x, y \in \mathcal{G}_0$ ,  $g \in \Gamma^x$ ,  $h \in \Gamma^y$ , and  $x \in \mathcal{G}_1$  with  $t(x) = x$ ,*
  - (a)  *$\rho(\gamma_g^x)a_h^y = 0$  for all  $a_h^y \in A_h^y$  with  $h \neq x^{-1}gx$ , and*
  - (b)  *$\rho(\gamma_g^x)$  is a vector space isomorphism from  $A_{x^{-1}gx}^{s(x)}$  to  $A_g^{t(x)}$ .*

*Proof.* Since

$$\gamma_g^{e_x} \cdot \gamma_h^{e_y} = \gamma_h^{e_y} \cdot \gamma_g^{e_x} = \delta_{g,h} \gamma_g^{e_x} \quad (34)$$

for  $g \in \Gamma^x$  and  $h \in \Gamma^y$ , it follows from (9) that

$$1^x \cdot 1^y = \sum_{g \in \Gamma^x} \sum_{h \in \Gamma^y} \delta_{g,h} \gamma_g^{e_x} = \delta_{x,y} \sum_{g \in \Gamma^x} \gamma_g^{e_x} = \delta_{x,y} 1^x. \quad (35)$$

Hence,

$$\rho(1^x) \circ \rho(1^y) = \delta_{x,y} \rho(1^x). \quad (36)$$

It follows from (36) and the definition of  $A^x$  that

$$\rho(1^y)a^x = \delta_{x,y} a^x \quad (37)$$

for  $a^x \in A^x$ .

Since  $\rho(1) = id_A$ , we also have

$$A = \rho(1)A = \sum_{x \in \mathcal{G}_0} \rho(1^x)A = \sum_{x \in \mathcal{G}_0} A^x. \quad (38)$$

In addition, if  $\sum_{x \in \mathcal{G}_0} a^x = 0$  for  $a^x \in A^x$ , it follows from (37) that

$$a^y = \rho(1^y) \left( \sum_{x \in \mathcal{G}_0} a^x \right) = 0 \quad (39)$$

for all  $y \in \mathcal{G}_0$ . (38) and (39) then show that  $A$  is a direct sum of the  $A^x$ 's. This completes the proof of (i).

For (ii), note that by (37)

$$A^x = \rho(1^x)A^x = \sum_{g \in \Gamma^x} \rho(\gamma_g^{e_x})A^x = \sum_{g \in \Gamma^x} A_g^x \quad (40)$$

and by (34)

$$\rho(\gamma_h^{e_y}) \circ \rho(\gamma_g^{e_x}) = \delta_{g,h} \rho(\gamma_g^{e_x}). \quad (41)$$

It follows from (41) and the definition of  $A_g^x$  that

$$\rho(\gamma_h^{e_y})a_g^x = \delta_{g,h} a_g^x \quad (42)$$

for  $a_g^x \in A_g^x$ . Using (40) and (42) and applying an argument similar to the one used to prove that  $A = \bigoplus_{x \in \mathcal{G}_0} A^x$  shows that  $A^x$  itself is a direct sum of the  $A_g^x$ 's. In addition, we also have

$$A_g^x := \rho(\gamma_g^{e_x})A^x = \rho(\gamma_g^{e_x})(\rho(1^x)A) = \rho(\gamma_g^{e_x})A, \quad (43)$$

where the second equality follows from the definition of  $A^x$  and the third equality follows from the fact that  $\gamma_g^{e_x} \cdot 1^x = \gamma_g^{e_x}$ . This completes the proof of (ii).

(iii-a) follows from (42) and the fact that

$$\begin{aligned} \rho(\gamma_g^x) &= \rho(\gamma_g^{e_{t(x)}} \cdot \gamma_g^x \cdot \gamma_{x^{-1}gx}^{e_{s(x)}}) \\ &= \rho(\gamma_g^{e_{t(x)}}) \circ \rho(\gamma_g^x) \circ \rho(\gamma_{x^{-1}gx}^{e_{s(x)}}). \end{aligned} \quad (44)$$

(44) also shows that  $\rho(\gamma_g^x)A_{x^{-1}gx}^{s(x)} \subset A_g^{t(x)}$ . To see that  $\rho(\gamma_g^x)$  is also an isomorphism, note that

$$\rho(\gamma_g^x) \circ \rho(\gamma_{x^{-1}gx}^{x^{-1}})|_{A_g^{t(x)}} = \rho(\gamma_g^x \cdot \gamma_{x^{-1}gx}^{x^{-1}})|_{A_g^{t(x)}} = \rho(\gamma_g^{e_{t(x)}})|_{A_g^{t(x)}} = id_{A_g^{t(x)}} \quad (45)$$

and

$$\rho(\gamma_{x^{-1}gx}^{x^{-1}}) \circ \rho(\gamma_g^x)|_{A_{x^{-1}gx}^{s(x)}} = \rho(\gamma_{x^{-1}gx}^{x^{-1}} \cdot \gamma_g^x)|_{A_{x^{-1}gx}^{s(x)}} = \rho(\gamma_{x^{-1}gx}^{e_{s(x)}})|_{A_{x^{-1}gx}^{s(x)}} = id_{A_{x^{-1}gx}^{s(x)}}. \quad (46)$$

This completes the proof of (iii-b).  $\square$

**Corollary 3.6.** *Every left  $D(k[\mathcal{G}])$ -module  $(\rho, A)$  has a left  $\mathcal{G}$ -action  $\varphi$  which acts as a  $k$ -linear map on the direct sum decomposition  $A = \bigoplus_{x \in \mathcal{G}_0} A^x$  given by part (i) of Proposition 3.5 where*

$$\varphi(x) := \sum_{g \in \Gamma^{t(x)}} \rho(\gamma_g^x)|_{A^{s(x)}} \text{ for } x \in \mathcal{G}_1. \quad (47)$$

*In addition, if  $x \in \mathcal{G}_1$  and  $A^{s(x)} = \bigoplus_{h \in \Gamma^{s(x)}} A_h^{s(x)}$  is the direct sum decomposition given by part (ii) of Proposition 3.5, then  $\varphi(x)A_g^{s(x)} \subset A_{xgx^{-1}}^{t(x)}$ .*

*Proof.* If  $y \in \mathcal{G}_1$  with  $t(y) = s(x)$ , then  $\varphi(x) \circ \varphi(y) = \varphi(xy)$  since

$$\sum_{g \in \Gamma^{t(x)}} \gamma_g^x \cdot \sum_{h \in \Gamma^{t(y)}} \gamma_h^y = \sum_{g \in \Gamma^{t(xy)}} \gamma_g^{xy}. \quad (48)$$

It follows from the definition of  $\varphi$  and part (iii) of Proposition 3.5 that  $\varphi(x)$  is a linear map from  $A^{s(x)}$  to  $A^{t(x)}$ .

Next we verify that  $\varphi(e_x) = id_{A^x}$ . To do this, let  $a^x \in A^x$ . Then  $a^x$  can be uniquely decomposed as  $a^x = \sum_{g \in \Gamma^x} a_g^x$  for  $a_g^x \in A_g^x$ . By part (ii) of Proposition 3.5, we have

$$\varphi(e_x)a^x = \sum_{g \in \Gamma^x} \rho(\gamma_g^{e_x})a^x = \sum_{g \in \Gamma^x} \rho(\gamma_g^{e_x})a_g^x = \sum_{g \in \Gamma^x} a_g^x = a^x. \quad (49)$$

To complete the proof that  $\varphi$  is a  $\mathcal{G}$ -action, we only need to show that  $\varphi(x) : A^{s(x)} \rightarrow A^{t(x)}$  is an isomorphism of vector spaces and this follows from the previous calculation since

$$\varphi(x^{-1}) \circ \varphi(x) = \varphi(x^{-1}x) = \varphi(e_{s(x)}) = id_{A^{s(x)}} \quad (50)$$

and

$$\varphi(x) \circ \varphi(x^{-1}) = \varphi(xx^{-1}) = \varphi(e_{t(x)}) = id_{A^{t(x)}}. \quad (51)$$

Lastly, if  $a_g^{s(x)} \in A_g^{s(x)}$ , then

$$\varphi(x)a_g^{s(x)} = \sum_{h \in \Gamma^{t(x)}} \rho(\gamma_h^x)a_g^{s(x)} = \rho(\gamma_{xgx^{-1}}^x)a_g^{s(x)} \in A_{xgx^{-1}}^{t(x)} \quad (52)$$

by part (iii) of Proposition 3.5.  $\square$

**Notation 3.7.** For an object  $(\rho, A)$  of  $\text{Rep}(D(k[\mathcal{G}]))$ , we will often suppress the  $D(k[\mathcal{G}])$ -action  $\rho$  and simply write  $A$  for  $(\rho, A)$ . The action of  $h \in D(k[\mathcal{G}])$  on  $a \in A$  will be denoted as  $h \triangleright a$  when  $\rho$  is omitted, that is,  $h \triangleright a := \rho(h)a$ . Furthermore, for  $x \in \mathcal{G}_0$  and  $g \in \Gamma^x$ ,  $A^x$  will denote the direct summand of  $A$  given by part (i) of Proposition 3.5, and  $A_g^x$  will denote the direct summand of  $A^x$  given by part (ii) of Proposition 3.5.

**Lemma 3.8.** *If  $A$  and  $B$  are objects of  $\text{Rep}(D(k[\mathcal{G}]))$ , then their monoidal product (with  $D(k[\mathcal{G}])$ -action induced by the coproduct  $\Delta_D$  of  $D(k[\mathcal{G}])$ ) is*

$$A \hat{\otimes} B = \bigoplus_{x \in \mathcal{G}_0} A^x \otimes_k B^x. \quad (53)$$

In addition, for  $y \in \mathcal{G}_0$

$$(A \hat{\otimes} B)^y = A^y \otimes_k B^y. \quad (54)$$

*Proof.* By (19),  $A \hat{\otimes} B := (1_{(1)} \triangleright A) \otimes_k (1_{(2)} \triangleright B)$ . It follows from (16) and part (i) of Proposition 3.5 that

$$A \hat{\otimes} B = \sum_{x \in \mathcal{G}_0} (1^x \triangleright A) \otimes_k (1^x \triangleright B) = \sum_{x \in \mathcal{G}_0} A^x \otimes_k B^x. \quad (55)$$

(53) then follows from the fact that  $A = \bigoplus_{x \in \mathcal{G}_0} A^x$  and  $B = \bigoplus_{x \in \mathcal{G}_0} B^x$ .

Lastly, note that since the  $D(k[\mathcal{G}])$ -action on  $A \hat{\otimes} B$  is induced by the coproduct of  $D(k[\mathcal{G}])$  and  $\Delta_D(1^y) = 1^y \otimes 1^y$  by (15), (54) follows readily from part (i) of Proposition 3.5 which implies that the image of  $A \hat{\otimes} B$  under  $1^y \otimes 1^y$  is  $A^y \otimes_k B^y$ .  $\square$

**Lemma 3.9.** (i) *The unit object  $D(k[\mathcal{G}])_t$  of  $\text{Rep}(D(k[\mathcal{G}]))$  is defined as follows:*

- (a) *As a vector space over  $k$ ,  $D(k[\mathcal{G}])_t$  has basis  $\{1^x\}_{x \in \mathcal{G}_0}$  where  $1^x \in D(k[\mathcal{G}])$  is defined by (9).*
- (b) *The left  $D(k[\mathcal{G}])$ -action on  $D(k[\mathcal{G}])_t$  is given by*

$$\gamma_h^y \triangleright 1^x = \delta_{s(y), x} \delta_{h, yy^{-1}} 1^{s(h)}. \quad (56)$$

- (ii)  *$D(k[\mathcal{G}])_t^x = k1^x$  for  $x \in \mathcal{G}_0$ . In particular,*

$$D(k[\mathcal{G}])_t \hat{\otimes} A = \bigoplus_{x \in \mathcal{G}_0} k1^x \otimes_k A^x \quad (57)$$

$$A \hat{\otimes} D(k[\mathcal{G}])_t = \bigoplus_{x \in \mathcal{G}_0} A^x \otimes_k k1^x \quad (58)$$

*for an object  $A$  of  $\text{Rep}(D(k[\mathcal{G}]))$ .*

- (iii) *If  $l_A : D(k[\mathcal{G}])_t \hat{\otimes} A \rightarrow A$  and  $r_A : A \hat{\otimes} D(k[\mathcal{G}])_t \rightarrow A$  are the left\right identity maps of  $\text{Rep}(D(k[\mathcal{G}]))$  for an object  $A$  of  $\text{Rep}(D(k[\mathcal{G}]))$ , then*

$$l_A(1^x \otimes a^x) = a^x \quad (59)$$

$$r_A(a^x \otimes 1^x) = a^x \quad (60)$$

*for  $a^x \in A^x$ .*

*Proof.* Let  $\varepsilon_D$  and  $\Delta_D$  denote the counit and coproduct of  $D(k[\mathcal{G}])$  respectively. By definition,  $D(k[\mathcal{G}])_t$  is the image of  $\varepsilon_{Dt} : D(k[\mathcal{G}]) \rightarrow D(k[\mathcal{G}])$  where  $\varepsilon_{Dt}$  is the map given by (20). Since

$$\Delta_D(1) = 1_{(1)} \otimes 1_{(2)} = \sum_{x \in \mathcal{G}_0} 1^x \otimes 1^x \quad (61)$$

by (16), we have

$$\begin{aligned} \varepsilon_{Dt}(h) &= \varepsilon_D(1_{(1)} \cdot h) 1_{(2)} \\ &= \sum_{x \in \mathcal{G}_0} \varepsilon_D(1^x \cdot h) 1^x \end{aligned} \quad (62)$$

for  $h \in D(k[\mathcal{G}])$ . Hence, the image of  $\varepsilon_t$  is contained in the subspace spanned by  $\{1^x\}_{x \in \mathcal{G}_0}$ . Since

$$\varepsilon_D(1^x \cdot 1^z) = \delta_{x,z} \varepsilon_D(1^x) = \delta_{x,z} \sum_{g \in \Gamma^x} \varepsilon_D(\gamma_g^{e_x}) = \delta_{x,z} \varepsilon_D(\gamma_{e_x}^{e_x}) = \delta_{x,z}, \quad (63)$$

we have  $\varepsilon_{Dt}(1^z) = 1^z$ . This shows that  $D(k[\mathcal{G}])_t$  is precisely the space spanned by  $\{1^x\}_{x \in \mathcal{G}_0}$ . It follows from (9) that the latter is also linearly independent and this completes the proof of (i-a).

For (i-b), we have

$$\begin{aligned}
 \gamma_h^y \triangleright 1^x &:= \varepsilon_{Dt}(\gamma_h^y \cdot 1^x) \\
 &= \delta_{s(y),x} \varepsilon_{Dt}(\gamma_h^y) \\
 &= \delta_{s(y),x} \sum_{z \in \mathcal{G}_0} \varepsilon_D(1^z \cdot \gamma_h^y) 1^z \\
 &= \delta_{s(y),x} \varepsilon_D(\gamma_h^y) 1^{s(h)} \\
 &= \delta_{s(y),x} \delta_{h,yy^{-1}} 1^{s(h)}.
 \end{aligned}$$

For (ii), note that if  $1^x \in D(k[\mathcal{G}])$  and  $1^y \in D(k[\mathcal{G}])_t$ , then by (i-b), we have

$$1^x \triangleright 1^y = \sum_{g \in \Gamma^x} \gamma_g^{\varepsilon_x} \triangleright 1^y = \sum_{g \in \Gamma^x} \delta_{x,y} \delta_{g,\varepsilon_x} 1^{s(g)} = \delta_{x,y} 1^x. \quad (64)$$

Hence, it follows from this and (i-a) that

$$D(k[\mathcal{G}])_t^x := 1^x \triangleright D(k[\mathcal{G}])_t = k 1^x. \quad (65)$$

(57) and (58) then follow from Lemma 3.8.

For (iii), we have

$$l_A(1^x \otimes a^x) = \sum_{y \in \mathcal{G}_0} l_A(1^y \triangleright 1^x \otimes 1^y \triangleright a^x) \quad (66)$$

$$= l_A(1_{(1)} \triangleright 1^x \otimes 1_{(2)} \triangleright a^x) \quad (67)$$

and

$$r_A(a^x \otimes 1^x) = \sum_{y \in \mathcal{G}_0} r_A(1^y \triangleright a^x \otimes 1^y \triangleright 1^x) \quad (68)$$

$$= r_A(1_{(1)} \triangleright a^x \otimes 1_{(2)} \triangleright 1^x) \quad (69)$$

where (66) and (68) follow from (37) and (64), and (67) and (69) follow from (16). By (25) and (27), we have

$$l_A(1_{(1)} \triangleright 1^x \otimes 1_{(2)} \triangleright a^x) = 1^x \triangleright a^x = a^x \quad (70)$$

and

$$r_A(1_{(1)} \triangleright a^x \otimes 1_{(2)} \triangleright 1^x) = S(1^x) \triangleright a^x = 1^x \triangleright a^x = a^x \quad (71)$$

which proves (iii).  $\square$

**Proposition 3.10.** *Suppose  $(A, m, \mu)$  is an algebra object in  $\text{Rep}(D(k[\mathcal{G}]))$  and*

$$a^x \bullet b^y := \begin{cases} m(a^x \otimes b^y) & \text{if } x = y \\ 0 & \text{if } x \neq y \end{cases} \quad (72)$$

for  $a^x \in A^x$ ,  $b^y \in A^y$  and

$$a \bullet b := \sum_{x,y \in \mathcal{G}_0} a^x \bullet b^y = \sum_{x \in \mathcal{G}_0} a^x \bullet b^x \quad (73)$$

for  $a = \sum_{x \in \mathcal{G}_0} a^x$ ,  $b = \sum_{x \in \mathcal{G}_0} b^x$  where  $a^x, b^x \in A^x$  for all  $x \in \mathcal{G}_0$ . Then

(i) for  $a_g^x \in A_g^x$  and  $b_h^x \in A_h^x$ ,

$$a_g^x \bullet b_h^x \in A_{gh}^x. \quad (74)$$

In particular,  $a^x \bullet b^y = \delta_{x,y} a^x \bullet b^y \in A^x$ ;

(ii)  $A$  is a unital associative algebra over  $k$  with multiplication  $\bullet$  and unit  $\mathbf{1}_A := \mu(1) = \sum_{x \in \mathcal{G}_0} \mu(1^x)$ ;

(iii) if  $(A, m, \mu)$  is also commutative, then

$$a_g^x \bullet b_h^x = (\varphi(g)b_h^x) \bullet a_g^x \quad (75)$$

where  $\varphi$  is the  $\mathcal{G}$ -action given by Corollary 3.6.

*Proof.* For (74), we have

$$a_g^x \bullet b_h^x = m(a_g^x \otimes b_h^x) \quad (76)$$

$$= m(\gamma_g^{e_x} \triangleright a_g^x \otimes \gamma_h^{e_x} \triangleright b_h^x) \quad (77)$$

$$= m \left( \sum_{\{g_1, h_1 \in \Gamma^x \mid g_1 h_1 = gh\}} \gamma_{g_1}^{e_x} \triangleright a_g^x \otimes \gamma_{h_1}^{e_x} \triangleright b_h^x \right) \quad (78)$$

$$= \gamma_{gh}^{e_x} \triangleright m(a_g^x \otimes b_h^x) \quad (79)$$

$$= \gamma_{gh}^{e_x} \triangleright (a_g^x \bullet b_h^x) \in A_{gh}^x \quad (80)$$

where (77), (78), and (80) follow from statement (ii) of Proposition 3.5 and (79) follows from the fact that

- (1)  $m$  is a  $D(k[\mathcal{G}])$ -linear map from  $A \hat{\otimes} A$  to  $A$  where  $A \hat{\otimes} A = \bigoplus_{y \in \mathcal{G}_0} A^y \otimes_k A^y$  by Lemma 3.8;
- (2) the  $D(k[\mathcal{G}])$ -action on  $A \hat{\otimes} A$  is induced by the coproduct of  $D(k[\mathcal{G}])$ ;
- and
- (3)  $\Delta(\gamma_{gh}^{e_x}) = \sum_{\{g_1, h_1 \in \Gamma^x \mid g_1 h_1 = gh\}} \gamma_{g_1}^{e_x} \otimes \gamma_{h_1}^{e_x}$ .

Since  $A^x = \bigoplus_{g \in \Gamma^x} A_g^x$ , it follows readily from (74) (and the definition of the product) that  $a^x \bullet b^y = \delta_{x,y} a^x \bullet b^y \in A^x$  for  $a^x \in A^x$ ,  $b^y \in A^y$ . This completes the proof of (i).

For (ii), note that since  $m$  is also  $k$ -linear, it follows that  $(\lambda_1 a) \bullet (\lambda_2 b) = (\lambda_1 \lambda_2)(a \bullet b)$  for  $\lambda_1, \lambda_2 \in k$  and  $a, b \in A$ . To show that  $\mathbf{1}_A$  is the unit element of  $A$ , we use the fact that  $m \circ (\mu \otimes id_A) = l_A$  and  $m \circ (id_A \otimes \mu) = r_A$  (where  $l_A$  and  $r_A$  denotes the left and right identity maps of  $A$ ). The latter shows that

$$m(\mu(1^x) \otimes a^x) = l_A(1^x \otimes a^x) = a^x \quad (81)$$

$$m(a^x \otimes \mu(1^x)) = r_A(a^x \otimes 1^x) = a^x \quad (82)$$

for  $a^x \in A^x$ , where part (iii) of Lemma 3.9 has been applied in (81) and (82). Now for  $a \in A$ ,  $a$  can be uniquely written as  $\sum_{x \in \mathcal{G}_0} a^x$  where  $a^x \in A^x$ .

By (81) and (82), we have the following:

$$\mathbf{1}_A \bullet a = \sum_{x \in \mathcal{G}_0} \mu(1^x) \bullet a^x = \sum_{x \in \mathcal{G}_0} m(\mu(1^x) \otimes a^x) = \sum_{x \in \mathcal{G}_0} a^x = a \quad (83)$$

$$a \bullet \mathbf{1}_A = \sum_{x \in \mathcal{G}_0} a^x \bullet \mu(1^x) = \sum_{x \in \mathcal{G}_0} m(a^x \otimes \mu(1^x)) = \sum_{x \in \mathcal{G}_0} a^x = a. \quad (84)$$

For associativity, it suffices to check that

$$(a^x \bullet b^y) \bullet c^z = a^x \bullet (b^y \bullet c^z) \quad (85)$$

for  $a^x \in A^x$ ,  $b^y \in A^y$ , and  $c^z \in A^z$ . From the definition of the product, it is easy to see that both sides of (85) are zero when  $x \neq y$  or  $y \neq z$ . For the case when  $x = y = z$ , we have

$$(a^x \bullet b^x) \bullet c^x = m(m(a^x \otimes b^x) \otimes c^x) \quad (86)$$

and

$$a^x \bullet (b^x \bullet c^x) = m(a^x \otimes m(b^x \otimes c^x)). \quad (87)$$

Since

$$m \circ (m \otimes id_A) = m \circ (id_A \otimes m) \quad (88)$$

we see that (86) and (87) are indeed equal. In addition, it follows readily from (72) and (73) that

$$a \bullet (b + c) = a \bullet b + a \bullet c \quad (89)$$

$$(b + c) \bullet a = b \bullet a + c \bullet a \quad (90)$$

for  $a, b, c \in A$ . This completes the proof of (ii).

Lastly for (iii), note that if  $(A, m, \mu)$  is also commutative, we have

$$a_g^x \bullet b_h^x = m(a_g^x \otimes b_h^x) \quad (91)$$

$$= m \circ c_{A,A}(a_g^x \otimes b_h^x) \quad (92)$$

$$= m \left( \sum_{y \in \mathcal{G}_0} \sum_{l, m \in \Gamma^y} (\gamma_m^l \triangleright b_h^x) \otimes (\gamma_l^{e_y} \triangleright a_g^x) \right) \quad (93)$$

$$= m \left( (\gamma_{ghg^{-1}}^g \triangleright b_h^x) \otimes (\gamma_g^{e_x} \triangleright a_g^x) \right) \quad (94)$$

$$= m \left( (\gamma_{ghg^{-1}}^g \triangleright b_h^x) \otimes a_g^x \right) \quad (95)$$

$$= (\gamma_{ghg^{-1}}^g \triangleright b_h^x) \bullet a_g^x \quad (96)$$

where the second equality follows from the fact that  $m = m \circ c_{A,A}$  (since  $(A, m, \mu)$  is a *commutative* algebra object); the third equality follows from the definition of the braiding morphism  $c$ , which is given by (29), and the definition of the R-matrix of  $D(k[\mathcal{G}])$ , which is given by (13) and (14); and the fourth and fifth equalities follow from parts (ii) and (iii) of Proposition 3.5. Since

$$\gamma_{ghg^{-1}}^g \triangleright b_h^x = \sum_{l \in \Gamma^x} \gamma_l^g \triangleright b_h^x \quad (97)$$

(by (iii-a) of Proposition 3.5) and the right side is just  $\varphi(g)b_h^x$ , we have

$$a_g^x \bullet b_h^x = (\varphi(g)b_h^x) \bullet a_g^x \quad (98)$$

and this completes the proof of Proposition 3.10.  $\square$

**Lemma 3.11.** *Let  $(A, \Delta, \varepsilon)$  be a coalgebra object in  $\text{Rep}(D(k[\mathcal{G}]))$ . Then  $\varepsilon : A \rightarrow D(k[\mathcal{G}])_t$  and  $\Delta : A \rightarrow A \hat{\otimes} A$  satisfy the following:*

(i)  $\varepsilon(a_g^x) = \delta_{g, e_x} \varepsilon(a_g^x) \in k1^x \subset D(k[\mathcal{G}])_t$ , and

(ii)  $\Delta(a_g^x) \in \bigoplus_{\{g_1, g_2 \in \Gamma^x \mid g_1 g_2 = g\}} A_{g_1}^x \otimes_k A_{g_2}^x$

for  $a_g^x \in A_g^x$ .

*Proof.* By part (i-a) of Lemma 3.9,  $\varepsilon(a_g^x)$  can be written as

$$\varepsilon(a_g^x) = \sum_{y \in \mathcal{G}_0} \lambda^y 1^y, \quad \lambda^y \in k. \quad (99)$$

Then

$$\varepsilon(a_g^x) = \varepsilon(\gamma_g^{e_x} \triangleright a_g^x) \quad (100)$$

$$= \gamma_g^{e_x} \triangleright \varepsilon(a_g^x) \quad (101)$$

$$= \sum_{y \in \mathcal{G}_0} \lambda^y \gamma_g^{e_x} \triangleright 1^y \quad (102)$$

$$= \sum_{y \in \mathcal{G}_0} \delta_{x,y} \delta_{g, e_x} \lambda^y 1^y \quad (103)$$

$$= \delta_{g, e_x} \lambda^x 1^x \quad (104)$$

which proves part (i) of Lemma 3.11. In the above calculation, the fourth equality follows from part (i-b) of Lemma 3.9.

For part (ii), we have

$$\Delta(a_g^x) = \Delta(\gamma_g^{e_x} \triangleright a_g^x) \quad (105)$$

$$= \sum_{g_1 g_2 = g} (\gamma_{g_1}^{e_x} \triangleright (a_g^x)_{(1)}) \otimes (\gamma_{g_2}^{e_x} \triangleright (a_g^x)_{(2)}) \in \bigoplus_{g_1 g_2 = g} A_{g_1}^x \otimes_k A_{g_2}^x \quad (106)$$

where the above calculation follows from part (ii) of Proposition 3.5 and the fact that  $\gamma_g^{e_x}$  acts on  $\Delta(a_g^x) = (a_g^x)_{(1)} \otimes (a_g^x)_{(2)}$  via the coproduct of  $D(k[\mathcal{G}])$ . (Note that the sum and direct sum in (106) are over all  $g_1, g_2 \in \Gamma^x$  such that  $g_1 g_2 = g$ .)  $\square$

**Proposition 3.12.** *Suppose  $(A, m, \mu)$  and  $(A, \Delta, \varepsilon)$  are respectively algebra and colagebra objects in  $\text{Rep}(D(k[\mathcal{G}]))$ . Let  $\varepsilon' : A \rightarrow k$  be the  $k$ -linear map defined by<sup>5</sup>*

$$\varepsilon(a) = \sum_{x \in \mathcal{G}_0} \varepsilon'(a^x) 1^x \quad (107)$$

---

<sup>5</sup>Note that by part (i) of Lemma 3.11 the coefficient of  $1^x$  in (107) depends only upon  $a^x$ , the  $x$ -component of  $a$ .



for  $a = \sum_{x \in \mathcal{G}_0} a^x$  with  $a^x \in A^x$  and let  $\eta : A \times A \rightarrow k$  be the map defined by

$$\eta(a, b) = \varepsilon'(a \bullet b) \quad (108)$$

where  $a \bullet b$  is the product given in Proposition 3.10. Then

- (i)  $\eta$  is a  $k$ -bilinear map;
- (ii)  $\eta(a \bullet b, c) = \eta(a, b \bullet c)$  for  $a, b, c \in A$
- (iii)  $\eta(a_g^x, b_h^x) = 0$  for  $gh \neq e_x$  where  $a_g^x \in A_g^x$  and  $b_h^x \in A_h^x$ ; and
- (iv)  $\eta(a^x, b^x) = \eta(\varphi(x)a^x, \varphi(x)b^x)$  where  $\varphi$  is the  $\mathcal{G}$ -action defined in Corollary 3.6,  $a^x, b^x \in A^x$ , and  $x \in \mathcal{G}_1$  with  $s(x) = x$ .

*Proof.* (i) follows easily from the definition of  $\eta$  and (ii) follows from the fact that  $(a \bullet b) \bullet c = a \bullet (b \bullet c)$  by Proposition 3.10.

For (iii), note that since  $a_g^x \bullet b_h^x \in A_{gh}^x$  (by Proposition 3.10), we have

$$\varepsilon(a_g^x \bullet b_h^x) = \varepsilon'(a_g^x \bullet b_h^x)1^x. \quad (109)$$

With  $a_g^x \bullet b_h^x \in A_{gh}^x$ , it follows from part (i) of Lemma 3.11 that  $\varepsilon(a_g^x \bullet b_h^x) = 0$  for  $gh \neq e_x$ . By (109),  $\varepsilon'(a_g^x \bullet b_h^x)$  is also zero for  $gh \neq e_x$  and this proves (iii).

For (iv), it suffices to consider the case when  $a^x = a_g^x \in A_g^x$  and  $b^x = b_h^x \in A_h^x$ . By Corollary 3.6,  $\varphi(x)a_g^x \in A_{xgx^{-1}}^y$  and  $\varphi(x)b_h^x \in A_{xhx^{-1}}^y$  where we have set  $y = t(x)$ . If  $gh \neq e_x$ , then

$$\eta(a_g^x, b_h^x) = \eta(\varphi(x)a_g^x, \varphi(x)b_h^x) = 0 \quad (110)$$

by part (iii) of Proposition 3.12.

For the case when  $gh = e_x$ , we have

$$\begin{aligned} \eta(\varphi(x)a_g^x, \varphi(x)b_{g^{-1}}^x) &= \varepsilon' \left( (\varphi(x)a_g^x) \bullet (\varphi(x)b_{g^{-1}}^x) \right) \\ &= \varepsilon' \left( (\gamma_{xgx^{-1}}^x \triangleright a_g^x) \bullet (\gamma_{xg^{-1}x^{-1}}^x \triangleright b_{g^{-1}}^x) \right) \\ &= \varepsilon' \left( m((\gamma_{xgx^{-1}}^x \triangleright a_g^x) \otimes (\gamma_{xg^{-1}x^{-1}}^x \triangleright b_{g^{-1}}^x)) \right) \\ &= \varepsilon' \left( m \left( \sum_{uv=e_y, u,v \in \Gamma^y} (\gamma_u^x \triangleright a_g^x) \otimes (\gamma_v^x \triangleright b_{g^{-1}}^x) \right) \right) \\ &= \varepsilon' \left( \gamma_{e_y}^x \triangleright m(a_g^x \otimes b_{g^{-1}}^x) \right) \\ &= \varepsilon' \left( \gamma_{e_y}^x \triangleright (a_g^x \bullet b_{g^{-1}}^x) \right) \end{aligned} \quad (111)$$

where the second equality follows from the definition of  $\varphi$  and part (iii-a) of Proposition 3.5; the fourth equality also follows from part (iii-a) of Proposition 3.5; and the fifth equality follows from the fact that  $m$  is  $D(k[\mathcal{G}])$ -linear.

In addition,

$$\varepsilon \left( \gamma_{e_y}^x \triangleright (a_g^x \bullet b_{g^{-1}}^x) \right) = \gamma_{e_y}^x \triangleright \varepsilon \left( a_g^x \bullet b_{g^{-1}}^x \right) \quad (112)$$

$$= \varepsilon' \left( a_g^x \bullet b_{g^{-1}}^x \right) \gamma_{e_y}^x \triangleright 1^x \quad (113)$$

$$= \varepsilon' \left( a_g^x \bullet b_{g^{-1}}^x \right) 1^y \quad (114)$$

where the third equality follows from part (i-b) of Lemma 3.9.

Since  $\gamma_{e_y}^x \triangleright (a_g^x \bullet b_{g^{-1}}^x) \in A_{e_y}^y$ , we also have

$$\varepsilon \left( \gamma_{e_y}^x \triangleright (a_g^x \bullet b_{g^{-1}}^x) \right) = \varepsilon' \left( \gamma_{e_y}^x \triangleright (a_g^x \bullet b_{g^{-1}}^x) \right) 1^y. \quad (115)$$

Hence,

$$\varepsilon' \left( \gamma_{e_y}^x \triangleright (a_g^x \bullet b_{g^{-1}}^x) \right) = \varepsilon' \left( a_g^x \bullet b_{g^{-1}}^x \right). \quad (116)$$

It follows from this as well as the definition of  $\eta$  and (111) that

$$\eta(\varphi(x)a_g^x, \varphi(x)b_{g^{-1}}^x) = \eta(a_g^x, b_{g^{-1}}^x) \quad (117)$$

and this completes the proof of (iv).  $\square$

**Proposition 3.13.** *Let  $(A, m, \Delta, \mu, \varepsilon)$  be a Frobenius object in  $\text{Rep}(D(k[\mathcal{G}]))$  and let  $\eta : A \times A \rightarrow k$  be the bilinear form given by Proposition 3.12. Then  $\eta|_{A_g^x \times A_h^x}$  is nondegenerate for all  $x \in \mathcal{G}_0$  and  $g, h \in \Gamma^x$  satisfying  $gh = e_x$ .*

*Proof.* To start, set

$$1_A^x := \mu(1^x) \in A^x. \quad (118)$$

Then from the proof of Proposition 3.10, we have

$$1_A^x \bullet a^x = a^x \bullet 1_A^x = a^x \quad (119)$$

for  $a^x \in A^x$ . Furthermore, (74) of Proposition 3.10 implies that  $1_A^x \in A_{e_x}^x$ . By (ii) of Lemma 3.11,

$$\Delta(1_A^x) \in \bigoplus_{h \in \Gamma^x} A_h^x \otimes_k A_{h^{-1}}^x. \quad (120)$$

Now let  $v_1, \dots, v_n$  be a basis for  $A^x$  with  $v_i \in A_{h_i}^x$  for some  $h_i \in \Gamma^x$ . Then

$$\Delta(1_A^x) = \sum_{i=1}^n u_i \otimes v_i \quad (121)$$

for some  $u_i \in A_{h_i^{-1}}^x$ .

If  $a^x \in A^x$ , then

$$a^x = l_A \circ (\varepsilon \otimes id_A) \circ \Delta(a^x) \quad (122)$$

$$= l_A \circ (\varepsilon \otimes id_A) \circ \Delta(a^x \bullet 1_A^x) \quad (123)$$

$$= l_A \circ (\varepsilon \otimes id_A) \circ \Delta \circ m(a^x \otimes 1_A^x) \quad (124)$$

$$= l_A \circ (\varepsilon \otimes id_A) \circ (m \otimes id_A) \circ (id_A \otimes \Delta)(a^x \otimes 1_A^x) \quad (125)$$

$$= \sum_{i=1}^n \varepsilon'(a^x \bullet u_i) v_i, \quad (126)$$

where the first equality is just the counit property of a coalgebra object; the fourth equality follows from (30); and the fifth equality employs the linear map  $\varepsilon' : A \rightarrow k$  that was defined in Proposition 3.12. Setting  $a^x = v_j$  and using the fact that the  $v_i$ 's are linearly independent gives

$$\eta(v_j, u_i) := \varepsilon'(v_j \bullet u_i) = \delta_{ji}. \quad (127)$$

Using (31), a similar calculation shows that

$$a^x = \sum_{i=1}^n \varepsilon'(v_i \bullet a^x) u_i. \quad (128)$$

Since  $a^x$  is arbitrary and the dimension of  $A^x$  is  $n$ , (128) shows that  $\{u_i\}_{i=1}^n$  is also a basis of  $A^x$ .

(127) combined with the fact that  $\{v_i\}_{i=1}^n$  and  $\{u_i\}_{i=1}^n$  are both bases of  $A^x$  (where  $v_i \in A_{h_i}^x$  and  $u_i \in A_{h_i^{-1}}^x$ ) shows that  $\eta|_{A_g^x \times A_h^x}$  is nondegenerate for  $gh = e_x$ .  $\square$

**Proposition 3.14.** *If  $((A, \rho), m, \Delta, \mu, \varepsilon)$  is a Frobenius object in  $\text{Rep}(D(k[\mathcal{G}]))$  which satisfies conditions (1) and (2) of Theorem 3.4, then*

$$< \mathcal{G}, (A, \bullet, \mathbf{1}_A), \eta, \varphi > \quad (129)$$

is a  $\mathcal{G}$ -FA where

- (i)  $\bullet$  and  $\mathbf{1}_A$  are respectively the product and multiplicative unit given by Proposition 3.10;
- (ii)  $\eta$  is the bilinear form given by Proposition 3.12; and
- (iii)  $\varphi$  is the  $\mathcal{G}$ -action given by Corollary 3.6.

*Proof.* Axioms (b), (i), and (ii) of Definition 3.1 are satisfied by parts (i) and (ii) of Proposition 3.5 and parts (i) and (ii) of Proposition 3.10.

Axioms (v) and (vii) of Definition 3.1 are satisfied by Corollary 3.6 and part (iii) of Proposition 3.10 respectively.

For axiom (d), we only need to verify that  $\varphi(x) : A^x \rightarrow A^y$  is an algebra isomorphism for  $x \in \mathcal{G}_1$  where  $s(x) = x$  and  $t(x) = y$ . By Corollary 3.6,  $\varphi(x)$  is already an isomorphism of vector spaces. Hence, we only need to check that

$$\varphi(x)(a^x \bullet b^x) = (\varphi(x)a^x) \bullet (\varphi(x)b^x). \quad (130)$$

It suffices to verify this for the case when  $a^x = a_g^x$  and  $b^x = b_h^x$ . In this case, we have

$$\varphi(x)(a_g^x \bullet b_h^x) = \gamma_{xghx^{-1}}^x \triangleright (a_g^x \bullet b_h^x) \quad (131)$$

$$= \gamma_{xghx^{-1}}^x \triangleright m(a_g^x \otimes b_h^x) \quad (132)$$

$$= m\left((\gamma_{xgx^{-1}}^x \triangleright a_g^x) \otimes (\gamma_{xhx^{-1}}^x \triangleright b_h^x)\right) \quad (133)$$

$$= m((\varphi(x)a_g^x) \otimes (\varphi(x)b_h^x)) \quad (134)$$

$$= (\varphi(x)a_g^x) \bullet (\varphi(x)b_h^x). \quad (135)$$

Throughout the above calculation we have made use of part (iii) of Proposition 3.5, and in the third equality, we have made use of the fact  $m$  is  $D(k[\mathcal{G}])$ -linear.

Axioms (c), (iii), and (iv) of Definition 3.1 are satisfied by Proposition 3.12; axiom (vi) of Definition 3.1 is satisfied by part (iii) of Proposition 3.12 and by Proposition 3.13.

All that remains left to do is to show that axioms (viii) and (ix) of Definition 3.1 are also satisfied. We will now show that axioms (viii) and (ix) follow respectively from conditions (1) and (2) of Theorem 3.4.

For axiom (viii) of Definition 3.1, let  $a_g^x \in A_g^x$ . Then

$$a_g^x = \sum_{y \in \mathcal{G}_0} \sum_{h \in \Gamma^y} \gamma_h^y \triangleright a_g^x \quad (136)$$

$$= \gamma_g^g \triangleright a_g^x \quad (137)$$

$$= \varphi(g)a_g^x \quad (138)$$

where the first equality follows from condition (1) of Theorem 3.4 and the second and third equalities follow from part (iii-a) of Proposition 3.5. This shows that axiom (viii) of Definition 3.1 is satisfied.

For axiom (ix) of Definition 3.1, let  $g, h \in A^x$  and for  $c \in A_{ghg^{-1}h^{-1}}^x$ , let  $l_c : A^x \rightarrow A^x$  be the linear map defined by  $l_c(a^x) := c \bullet a^x$ . Then by part (iii) of Proposition 3.5 and by part (i) of Proposition 3.10, we have the following:

$$\text{Tr}\left(l_c \circ \rho(\gamma_{hgh^{-1}}^h)\right) = \text{Tr}\left(l_c \circ \rho(\gamma_{hgh^{-1}}^h)|_{A_g^x} : A_g^x \rightarrow A_g^x\right) \quad (139)$$

$$\text{Tr}\left(\rho(\gamma_h^{g^{-1}}) \circ l_c \circ \rho(\gamma_h^{e_x})\right) = \text{Tr}\left(\rho(\gamma_h^{g^{-1}}) \circ l_c|_{A_h^x} : A_h^x \rightarrow A_h^x\right). \quad (140)$$

Condition (2) of Theorem 3.4 then gives

$$\text{Tr}\left(l_c \circ \rho(\gamma_{hgh^{-1}}^h)|_{A_g^x} : A_g^x \rightarrow A_g^x\right) = \text{Tr}\left(\rho(\gamma_h^{g^{-1}}) \circ l_c|_{A_h^x} : A_h^x \rightarrow A_h^x\right). \quad (141)$$

Since

$$l_c \circ \rho(\gamma_{hgh^{-1}}^h)|_{A_g^x} = l_c \circ \varphi(h)|_{A_g^x} \quad (142)$$

and

$$\rho(\gamma_h^{g^{-1}}) \circ l_c|_{A_h^x} = \varphi(g^{-1}) \circ l_c|_{A_h^x} \quad (143)$$

by part (iii) of Proposition 3.5 and the definition of  $\varphi$ , (141) shows that axiom (ix) is satisfied and this completes the proof of Proposition 3.14.  $\square$

**3.2. Frobenius objects via  $\mathcal{G}$ -FAs.** In this section, we move in the opposite direction and show that every  $\mathcal{G}$ -FA is also a Frobenius object in  $\text{Rep}(D(k[\mathcal{G}]))$  which satisfies conditions (1) and (2) of Theorem 3.4. We begin with the following result:

**Proposition 3.15.** *Every  $\mathcal{G}$ -FA is a left  $D(k[\mathcal{G}])$ -module. If  $A$  is a  $\mathcal{G}$ -FA with  $\mathcal{G}$ -action  $\varphi$ , then its  $D(k[\mathcal{G}])$ -action is the linear map defined by*

$$\rho(\gamma_g^x) a_h^y := \delta_{h, x^{-1}gx} \varphi(x) a_h^y \quad (144)$$

for  $\gamma_g^x \in D(k[\mathcal{G}])$  and  $a_h^y \in A_h^y$ .

*Proof.* To start, let  $a \in A$  and decompose it as  $a = \sum_{x \in \mathcal{G}_0} \sum_{g \in \Gamma^x} a_g^x$ . To show that (144) does indeed define a  $D(k[\mathcal{G}])$ -action, we need to verify that

- (i)  $\rho(1) = id_A$ , and
- (ii)  $\rho(\gamma_{g_1}^{x_1}) \circ \rho(\gamma_{g_2}^{x_2}) = \rho(\gamma_{g_1}^{x_1} \cdot \gamma_{g_2}^{x_2})$ .

For (i), we have

$$\rho(1)a = \sum_{x \in \mathcal{G}_0} \sum_{g \in \Gamma^x} \rho(\gamma_g^{e_x}) a = \sum_{x \in \mathcal{G}_0} \sum_{g \in \Gamma^x} \varphi(e_x) a_g^x = \sum_{x \in \mathcal{G}_0} \sum_{g \in \Gamma^x} a_g^x = a.$$

For (ii), we have

$$\begin{aligned} \rho(\gamma_{g_1}^{x_1}) \circ \rho(\gamma_{g_2}^{x_2}) a &= \rho(\gamma_{g_1}^{x_1}) \circ \varphi(x_2) a_{x_2^{-1}g_2x_2}^{s(x_2)} \\ &= \delta_{x_1^{-1}g_1x_1, g_2} \varphi(x_1) \circ \varphi(x_2) a_{x_2^{-1}g_2x_2}^{s(x_2)}. \end{aligned} \quad (145)$$

Since  $\gamma_{g_1}^{x_1} \cdot \gamma_{g_2}^{x_2} = \delta_{x_1^{-1}g_1x_1, g_2} \gamma_{g_1}^{x_1x_2}$ , we see that (ii) is satisfied for the case when  $x_1^{-1}g_1x_1 \neq g_2$ . For the case when  $x_1^{-1}g_1x_1 = g_2$ , (145) reduces to

$$\varphi(x_1x_2) a_{x_2^{-1}g_2x_2}^{s(x_2)} = \rho(\gamma_{x_1g_2x_1^{-1}}^{x_1x_2}) a_{x_2^{-1}g_2x_2}^{s(x_2)} \quad (146)$$

$$= \rho(\gamma_{g_1}^{x_1x_2}) a_{x_2^{-1}g_2x_2}^{s(x_2)} \quad (147)$$

$$= \rho(\gamma_{g_1}^{x_1x_2}) a \quad (148)$$

$$= \rho(\gamma_{g_1}^{x_1} \cdot \gamma_{g_2}^{x_2}) a \quad (149)$$

and this completes the proof of (ii).  $\square$

Proposition 3.15 will be applied implicitly throughout this section.

**Remark 3.16.** Note that if one applies (i) and (ii) of Proposition 3.5 to the left  $D(k[\mathcal{G}])$ -module given by Proposition 3.15, the resulting direct sum decomposition is exactly the one from the original  $\mathcal{G}$ -FA. Hence, if  $(\rho, A)$  is the left  $D(k[\mathcal{G}])$ -module of Proposition 3.15 and  $A = \bigoplus_{x \in \mathcal{G}_0} A^x$  is the direct sum decomposition of the original  $\mathcal{G}$ -FA, then the monoidal product  $A \hat{\otimes} A$  of  $(\rho, A)$  with itself is  $\bigoplus_{x \in \mathcal{G}_0} A^x \otimes_k A^x$  by Lemma 3.8.

**Proposition 3.17.** *If  $\langle \mathcal{G}, (A, \bullet, \mathbf{1}_A), \eta, \varphi \rangle$  is a  $\mathcal{G}$ -FA, then  $((\rho, A), m, \mu)$  is a commutative algebra object in  $\text{Rep}(D(k[\mathcal{G}]))$  where*

- (i)  $m : A \hat{\otimes} A \rightarrow A$  is given by  $m(a^x \otimes b^x) := a^x \bullet b^x \in A^x$ , and
- (ii)  $\mu : D(k[\mathcal{G}])_t \rightarrow A$  is given by  $1^x \mapsto \mathbf{1}_A^x := \rho(1^x) \mathbf{1}_A \in A_{e_x}^x$ .

*Proof.* Its clear from the associativity of the  $\mathcal{G}$ -FA product and the fact that  $\mathbf{1}_A$  is the multiplicative unit that  $m$  and  $\mu$  satisfy 1 and 2 of Definition 2.8.

Next, we verify that  $m \circ c_{A,A} = m$ . Without loss of generality, take  $a = a_g^x \in A_g^x$  and  $b = b_h^x \in A_h^x$ . Then

$$m(a_g^x \otimes b_h^x) = a_g^x \bullet b_h^x \quad (150)$$

$$= (\varphi(g)b_h^x) \bullet a_g^x \quad (151)$$

$$= m(\varphi(g)b_h^x \otimes a_g^x) \quad (152)$$

$$= m(\rho(\gamma_{ghg^{-1}}^g)b_h^x \otimes a_g^x) \quad (153)$$

$$= m(\rho(\gamma_{ghg^{-1}}^g)b_h^x \otimes \rho(\gamma_g^{e_x})a_g^x) \quad (154)$$

$$= m \left( \sum_{y \in \mathcal{G}_0} \sum_{l, m \in \Gamma^y} \rho(\gamma_m^l)b_h^x \otimes \rho(\gamma_l^{e_y})a_g^x \right) \quad (155)$$

$$= m \circ c_{A,A}(a_g^x \otimes b_h^x) \quad (156)$$

where the second equality follows from axiom (vii) of Definition 3.1.

The only thing that remains to be done is to show that  $m$  and  $\mu$  are  $D(k[\mathcal{G}])$ -linear. In the case of  $m$ , for  $x \in \mathcal{G}_1$  with  $s(x) = x$ , we have

$$\rho(\gamma_{xghx^{-1}}^x)m(a_g^x \otimes b_h^x) = \varphi(x)(a_g^x \bullet b_h^x) \quad (157)$$

$$= (\varphi(x)a_g^x) \bullet (\varphi(x)b_h^x) \quad (158)$$

$$= m \left( (\rho(\gamma_{xgx^{-1}}^x)a_g^x) \otimes (\rho(\gamma_{xhx^{-1}}^x)b_h^x) \right) \quad (159)$$

$$= \sum_{g_1 h_1 = gh} m \left( (\rho(\gamma_{xg_1 x^{-1}}^x)a_g^x) \otimes (\rho(\gamma_{xh_1 x^{-1}}^x)b_h^x) \right) \quad (160)$$

(where the sum in the last equality is over all  $g_1, h_1 \in \Gamma^x$  satisfying  $g_1 h_1 = gh$ ). Since the  $D(k[\mathcal{G}])$ -action on  $A \hat{\otimes} A$  is induced by the coproduct of  $D(k[\mathcal{G}])$ , the above calculation shows that  $m$  is  $D(k[\mathcal{G}])$ -linear.

In the case of  $\mu$ , it suffices to show that

$$\mu(\gamma_h^y \triangleright 1^x) = \rho(\gamma_h^y)\mu(1^x). \quad (161)$$

By (i-b) of Lemma 3.9, the left side is

$$\mu(\gamma_h^y \triangleright 1^x) = \delta_{s(y), x} \delta_{h, yy^{-1}} \mu(1^{s(h)}) = \delta_{s(y), x} \delta_{h, yy^{-1}} \mathbf{1}_A^{s(h)}, \quad (162)$$

and the right side is

$$\rho(\gamma_h^y)\mu(1^x) = \rho(\gamma_h^y)\rho(1^x)\mathbf{1}_A = \delta_{s(y), x}\rho(\gamma_h^y)\mathbf{1}_A. \quad (163)$$

Since

$$\mathbf{1}_A = \sum_{z \in \mathcal{G}_0} \mathbf{1}_A^z,$$

it follows easily from axioms (i) and (ii) of Definition 3.1 and the definition of  $\rho$  that  $\mathbf{1}_A^z$  is the unit element of  $A^z$  and  $\mathbf{1}_A^z \in A_{e_z}^z$ . Hence,

$$\begin{aligned}
 \rho(\gamma_h^y)\mathbf{1}_A &= \sum_{z \in \mathcal{G}_0} \rho(\gamma_h^y)\mathbf{1}_A^z \\
 &= \sum_{z \in \mathcal{G}_0} \delta_{e_z, y^{-1}hy} \varphi(y)\mathbf{1}_A^z \\
 &= \delta_{h, yy^{-1}} \varphi(y)\mathbf{1}_A^{s(y)} \\
 &= \delta_{h, yy^{-1}} \mathbf{1}_A^{t(y)}
 \end{aligned} \tag{164}$$

where the last equality follows from the fact that  $\varphi(y) : A^{s(y)} \rightarrow A^{t(y)}$  is an isomorphism of algebras and must therefore map the unit of  $A^{s(y)}$  to that of  $A^{t(y)}$ . By substituting (164) into (163) and using the fact that  $s(h) = t(y)$ , we see that the right side and left side of (161) are indeed equal and this completes the proof.  $\square$

**Notation 3.18.** As in the proof of Proposition 3.17, we will use  $\mathbf{1}_A^x$  to denote the unit element of  $A^x$ .

**Notation 3.19.** For a vector space  $V$ , let  $V^*$  denote the dual space of  $V$ , and for a linear map  $f : V \rightarrow U$ , let  $f^* : U^* \rightarrow V^*$  denote the dual of  $f$ .

**Lemma 3.20.** Suppose  $\langle \mathcal{G}, (A, \bullet, \mathbf{1}_A), \eta, \varphi \rangle$  is a  $\mathcal{G}$ -FA and  $\psi : A \rightarrow A^*$  is the  $k$ -linear map defined by

$$\psi(a)(b) := \eta(a, b)$$

where  $a, b \in A$  and  $\psi(a) \in A^*$ . Then

- (i)  $\psi|_{A_g^x}$  is a vector space isomorphism from  $A_g^x$  to  $(A_{g^{-1}}^x)^*$ , where an element  $f$  in  $(A_{g^{-1}}^x)^*$  is also regarded as an element in  $A^*$  via  $f(b_h^y) = \delta_{g^{-1}, h} f(b_h^y)$  for  $b_h^y \in A_h^y$ .
- (ii)  $\psi : A \rightarrow A^*$  is a vector space isomorphism; and
- (iii)  $\psi\left(\rho(\gamma_g^x)a_{x^{-1}gx}^x\right) = \rho(\gamma_{x^{-1}g^{-1}x}^{x^{-1}})^*\psi(a_{x^{-1}gx}^x)$ .

*Proof.* For (i), the isomorphism from  $A_g^x$  to  $(A_{g^{-1}}^x)^*$  follows directly from axiom (vi) of Definition 3.1. The same axiom also implies that  $\psi(a_g^x)(b_h^y) = 0$  when  $y = x$  and  $h \neq g^{-1}$ . For  $y \neq x$ , we have

$$\begin{aligned}
 \psi(a_g^x)(b_h^y) &= \eta(a_g^x, b_h^y) \\
 &= \eta(a_g^x \bullet b_h^y, \mathbf{1}_A) \\
 &= 0
 \end{aligned}$$

where the second and third equality follow from axioms (iii) and (ii) of Definition 3.1 respectively. In other words,

$$\psi(a_g^x)(b_h^y) = \delta_{g^{-1}, h} \psi(a_g^x)(b_h^y). \tag{165}$$

(ii) is a consequence of part (i) of Lemma 3.20 and the fact that  $A$  decomposes as  $A = \bigoplus_{x \in \mathcal{G}_0} \bigoplus_{g \in \Gamma^x} A_g^x$ .

For (iii), let  $b_h^y \in A_h^y$ . Then we need to show that

$$\psi \left( \rho(\gamma_g^x) a_{x^{-1}gx}^x \right) (b_h^y) = \psi(a_{x^{-1}gx}^x) \left( \rho(\gamma_{x^{-1}g^{-1}x}^{x^{-1}}) b_h^y \right). \quad (166)$$

Both sides of (166) are zero for the case when  $h \neq g^{-1}$  by (165) and the definition of  $\rho$ . For the case when  $h = g^{-1}$ , we have  $y = t(x)$  and

$$\begin{aligned} \psi \left( \rho(\gamma_g^x) a_{x^{-1}gx}^x \right) (b_{g^{-1}}^y) &= \eta(\rho(\gamma_g^x) a_{x^{-1}gx}^x, b_{g^{-1}}^y) \\ &= \eta(\varphi(x) a_{x^{-1}gx}^x, b_{g^{-1}}^y) \\ &= \eta(\varphi(x^{-1}) \varphi(x) a_{x^{-1}gx}^x, \varphi(x^{-1}) b_{g^{-1}}^y) \\ &= \eta(a_{x^{-1}gx}^x, \varphi(x^{-1}) b_{g^{-1}}^y) \\ &= \eta(a_{x^{-1}gx}^x, \rho(\gamma_{x^{-1}g^{-1}x}^{x^{-1}}) b_{g^{-1}}^y) \\ &= \psi(a_{x^{-1}gx}^x) \left( \rho(\gamma_{x^{-1}g^{-1}x}^{x^{-1}}) b_{g^{-1}}^y \right) \end{aligned}$$

where the third equality follows from axiom (iv) of Definition 3.1.  $\square$

**Lemma 3.21.** *Suppose  $\langle \mathcal{G}, (A, \bullet, \mathbf{1}_A), \eta, \varphi \rangle$  is a  $\mathcal{G}$ -FA and  $\varepsilon : A \rightarrow D(k[\mathcal{G}])_t$  is the  $k$ -linear map defined by*

$$\varepsilon(a) := \sum_{x \in \mathcal{G}_0} \eta(a^x, \mathbf{1}_A) 1^x \quad (167)$$

for  $a = \sum_{x \in \mathcal{G}_0} a^x$ . Then  $\varepsilon$  is  $D(k[\mathcal{G}])$ -linear.

*Proof.* It suffices to show that

$$\varepsilon \left( \rho(\gamma_g^x) a_h^y \right) = \gamma_g^x \triangleright \varepsilon(a_h^y) \quad (168)$$

for  $a_h^y \in A_h^y$ .

From the definition of  $\rho$ , we see that the left side of (168) is zero when  $y \neq s(x)$ . Likewise, the right side is also zero when  $y \neq s(x)$  since

$$\begin{aligned} \gamma_g^x \triangleright \varepsilon(a_h^y) &= \eta(a_h^y, \mathbf{1}_A) \gamma_g^x \triangleright 1^y \\ &= \delta_{s(x), y} \delta_{g, xx^{-1}} \eta(a_h^y, \mathbf{1}_A) 1^{t(x)} \\ &= \delta_{s(x), y} \delta_{g, xx^{-1}} \eta(a_h^y \bullet \mathbf{1}_A^y, \mathbf{1}_A) 1^{t(x)} \\ &= \delta_{s(x), y} \delta_{g, xx^{-1}} \eta(a_h^y, \mathbf{1}_A^y) 1^{t(x)} \end{aligned} \quad (169)$$

where the second equality follows from part (i-b) of Lemma 3.9 and the last equality follows from axiom (iii) of Definition 3.1.



For the case when  $y = s(x)$ , the left side of (168) can be rewritten as

$$\begin{aligned}
\varepsilon \left( \rho(\gamma_g^x) a_h^{s(x)} \right) &= \eta(\rho(\gamma_g^x) a_h^{s(x)}, \mathbf{1}_A) 1^{t(x)} \\
&= \delta_{x^{-1}gx, h} \eta(\varphi(x) a_h^{s(x)}, \mathbf{1}_A) 1^{t(x)} \\
&= \delta_{x^{-1}gx, h} \eta((\varphi(x) a_h^{s(x)}) \bullet \mathbf{1}_A^{t(x)}, \mathbf{1}_A) 1^{t(x)} \\
&= \delta_{x^{-1}gx, h} \eta(\varphi(x) a_h^{s(x)}, \mathbf{1}_A^{t(x)}) 1^{t(x)} \\
&= \delta_{x^{-1}gx, h} \eta(\varphi(x) a_h^{s(x)}, \varphi(x) \mathbf{1}_A^{s(x)}) 1^{t(x)} \\
&= \delta_{x^{-1}gx, h} \eta(a_h^{s(x)}, \mathbf{1}_A^{s(x)}) 1^{t(x)} \\
&= \delta_{g, xx^{-1}} \eta(a_h^{s(x)}, \mathbf{1}_A^{s(x)}) 1^{t(x)}
\end{aligned} \tag{170}$$

where the first equality follows from the fact that  $\rho(\gamma_g^x) a_h^{s(x)} \in A^{t(x)}$ ; the sixth equality follows from axiom (iv) of Definition 3.1; and the seventh equality follows from axiom (vi) of Definition 3.1 and the fact that  $\mathbf{1}_A^{s(x)} \in A_{e_{s(x)}}^{s(x)}$ .

By comparing (170) with (169), we see that (168) is also satisfied when  $y = s(x)$ .  $\square$

**Lemma 3.22.** *Suppose  $\langle \mathcal{G}, (A, \bullet, \mathbf{1}_A), \eta, \varphi \rangle$  is a  $\mathcal{G}$ -FA. Let  $\psi$  be the map given in Lemma 3.20,  $m^{op} : A \widehat{\otimes} A \rightarrow A$  be the  $k$ -linear map given by  $m^{op}(a^x \otimes b^x) := b^x \bullet a^x$ , and let  $\Delta : A \rightarrow A \widehat{\otimes} A$  be the  $k$ -linear map given by*

$$\Delta := (\psi^{-1} \otimes \psi^{-1}) \circ (m^{op})^* \circ \psi.$$

Then

- (i)  $\Delta(a_g^x) \in \bigoplus_{g_1 g_2 = g} A_{g_1}^x \otimes_k A_{g_2}^x$  for all  $a_g^x \in A_g^x$ , and
- (ii)  $\Delta$  is  $D(k[\mathcal{G}])$ -linear

(where the direct sum in (i) is over all  $g_1, g_2 \in \Gamma^x$  satisfying  $g_1 g_2 = g$ ).

*Proof.* For (i), note that by part (i) of Lemma 3.20,  $(m^{op})^* \circ \psi(a_g^x)(b_h^y \otimes c_l^y) = 0$  for all  $b_h^y \in A_h^y$  and  $c_l^y \in A_l^y$  satisfying  $lh \neq g^{-1}$ . This implies that

$$(m^{op})^* \circ \psi(a_g^x) \in \bigoplus_{g_1 g_2 = g} (A_{g_1}^x)^* \otimes (A_{g_2}^x)^*. \tag{171}$$

Part (i) of Lemma 3.22 then follows from part (i) of Lemma 3.20.

For (ii), it suffices to show that

$$\Delta(\rho(\gamma_g^x) a_h^y) = \sum_{g_1 g_2 = g} [\rho(\gamma_{g_1}^x) \otimes \rho(\gamma_{g_2}^x)] \Delta(a_h^y) \tag{172}$$

for  $a_h^y \in A_h^y$  (where the sum is over all  $g_1, g_2 \in \Gamma^{t(x)}$  satisfying  $g_1 g_2 = g$ ). From the definition of  $\rho$ , the left side is zero for  $h \neq x^{-1}gx$ . By (i) of Lemma 3.22, the right side is also zero for  $h \neq x^{-1}gx$ .

Let  $x = s(x)$ . For the case when  $h = x^{-1}gx$ , we have

$$\begin{aligned} (m^{op})^* \circ \psi(\rho(\gamma_g^x) a_{x^{-1}gx}^x) &= (m^{op})^* \circ \rho(\gamma_{x^{-1}gx}^{x^{-1}})^* \circ \psi(a_{x^{-1}gx}^x) \\ &= \left[ \sum_{g_1 g_2 = g} \rho(\gamma_{x^{-1}g_1^{-1}x}^{x^{-1}})^* \otimes \rho(\gamma_{x^{-1}g_2^{-1}x}^{x^{-1}})^* \right] \circ (m^{op})^* \circ \psi(a_{x^{-1}gx}^x) \end{aligned} \quad (173)$$

where the first equality follows from part (iii) of Lemma 3.20 and the second equality follows the definition of  $\rho$  and the fact that  $\varphi(x^{-1})$  is an algebra homomorphism.

Since

$$(m^{op})^* \circ \psi(a_{x^{-1}gx}^x) \in \bigoplus_{g_1 g_2 = g} (A_{x^{-1}g_1^{-1}x}^x)^* \otimes_k (A_{x^{-1}g_2^{-1}x}^x)^*$$

by (171), it follows from (i) of Lemma 3.20 that

$$(m^{op})^* \circ \psi(a_{x^{-1}gx}^x) = \sum_{g_1 g_2 = g} \sum_{i=1}^{n[g_1]} \psi(u_{x^{-1}g_1 x, i}^x) \otimes \psi(v_{x^{-1}g_2 x, i}^x) \quad (174)$$

for some  $u_{x^{-1}g_1 x, i}^x \in A_{x^{-1}g_1 x}^x$  and  $v_{x^{-1}g_2 x, i}^x \in A_{x^{-1}g_2 x}^x$ . In particular,

$$\Delta(a_{x^{-1}gx}^x) = \sum_{g_1 g_2 = g} \sum_{i=1}^{n[g_1]} u_{x^{-1}g_1 x, i}^x \otimes v_{x^{-1}g_2 x, i}^x. \quad (175)$$

Substituting (174) into the right side of (173) and applying (iii) of Lemma 3.20 as well as the fact that  $\rho(\gamma_s^y)^* \psi(c_t^z) = 0$  for  $s \neq t^{-1}$  gives

$$(m^{op})^* \circ \psi(\rho(\gamma_g^x) a_{x^{-1}gx}^x) = \sum_{g_1 g_2 = g} \sum_{i=1}^{n[g_1]} \psi(\rho(\gamma_{g_1}^x) u_{x^{-1}g_1 x, i}^x) \otimes \psi(\rho(\gamma_{g_2}^x) v_{x^{-1}g_2 x, i}^x). \quad (176)$$

Applying  $\psi^{-1} \otimes \psi^{-1}$  to both sides of (176) (and using the definition of  $\rho$ ) yields

$$\Delta(\rho(\gamma_g^x) a_{x^{-1}gx}^x) = \sum_{g_1 g_2 = g} [\rho(\gamma_{g_1}^x) \otimes \rho(\gamma_{g_2}^x)] \Delta(a_{x^{-1}gx}^x)$$

which completes the proof.  $\square$

**Proposition 3.23.** *Suppose  $\langle \mathcal{G}, (A, \bullet, 1_A), \eta, \varphi \rangle$  is a  $\mathcal{G}$ -FA and  $\varepsilon$  and  $\Delta$  are the maps given in Lemmas 3.21 and 3.22 respectively. Then  $((\rho, A), \Delta, \varepsilon)$  is a co-commutative coalgebra object in  $\text{Rep}(D(k[\mathcal{G}]))$ .*

*Proof.* By Lemmas 3.21 and 3.22,  $\varepsilon$  and  $\Delta$  are  $D(k[\mathcal{G}])$ -linear. We now verify that  $\Delta$  and  $\varepsilon$  satisfy the axioms of a co-commutative coalgebra.

For the coassociativity of  $\Delta$ , we have

$$(\Delta \otimes id_A) \circ \Delta = [[(\psi^{-1} \otimes \psi^{-1}) \circ (m^{op})^*] \otimes \psi^{-1}] \circ (m^{op})^* \circ \psi \quad (177)$$

$$= [\psi^{-1} \otimes \psi^{-1} \otimes \psi^{-1}] \circ [((m^{op})^* \otimes id_A) \circ (m^{op})^*] \circ \psi \quad (178)$$

$$= [\psi^{-1} \otimes \psi^{-1} \otimes \psi^{-1}] \circ [(id_A \otimes (m^{op})^*) \circ (m^{op})^*] \circ \psi \quad (179)$$

$$= [\psi^{-1} \otimes [(\psi^{-1} \otimes \psi^{-1}) \circ (m^{op})^*]] \circ (m^{op})^* \circ \psi \quad (180)$$

$$= (id_A \otimes \Delta) \circ \Delta \quad (181)$$

where the third equality is a consequence of the fact that the opposite multiplication map  $m^{op}$  of Lemma 3.22 is associative.

For the counit property, we need to show that

$$l_A \circ (\varepsilon \otimes id_A) \circ \Delta(a) = a = r_A \circ (id_A \otimes \varepsilon) \circ \Delta(a) \quad (182)$$

for all  $a \in A$ . By linearity, it suffices to prove (182) for the case when  $a = a_g^x \in A_g^x$ . If  $a_g^x$  is zero, there is nothing to prove. So assume then that  $a_g^x \neq 0$  and let  $\{u_j\}_{j=1}^n$  be a basis for  $A_{g^{-1}}^x$  and let  $\{v_i\}_{i=1}^m$  be a basis for  $A_{e_x}^x$  where  $v_1$  is taken to be the projection of  $\mathbf{1}_A$  onto  $A^x$ . (As was shown in proof of Proposition 3.17,  $v_1$  is indeed an element of  $A_{e_x}^x$  and is also the unit element of  $A^x$ .) Furthermore, let  $\{u_j^*\}_{j=1}^n$  and  $\{v_i^*\}_{i=1}^m$  denote the dual basis of  $\{u_j\}_{j=1}^n$  and  $\{v_i\}_{i=1}^m$  respectively (where an element  $f$  in  $(A_h^y)^*$  is also regarded as an element of  $A^*$  by extending the definition of  $f$  via  $f(a_l^z) = \delta_{h,l} f(a_l^z)$ ).

By part (i) of Lemma 3.20, we have

$$\psi(a_g^x) = \sum_{j=1}^n \alpha_j u_j^* \quad (183)$$

where  $\alpha_j = \psi(a_g^x)(u_j)$ . In addition, by part (i) of Lemmas 3.20 and 3.22 we can also express  $(m^{op})^* \circ \psi(a_g^x)$  as

$$(m^{op})^* \circ \psi(a_g^x) = \sum_{i,j} \alpha_{ij} v_i^* \otimes u_j^* + \omega \in \bigoplus_{g_1 g_2 = g} (A_{g_1}^x)^* \otimes_k (A_{g_2}^x)^* \quad (184)$$

where  $\alpha_{ij} = \psi(a_g^x)(u_j \bullet v_i)$  and

$$\omega \in \bigoplus_{g_1 g_2 = g, g_1 \neq e_x} (A_{g_1}^x)^* \otimes_k (A_{g_2}^x)^*.$$

In particular, note that  $\alpha_j = \alpha_{1j}$ .

Next, note that for  $f \in (A_h^x)^*$ , we have

$$\begin{aligned} \varepsilon \circ \psi^{-1}(f) &= \eta(\psi^{-1}(f), \mathbf{1}_A) \mathbf{1}^x \\ &= \psi(\psi^{-1}(f))(\mathbf{1}_A) \mathbf{1}^x \\ &= f(\mathbf{1}_A) \mathbf{1}^x \\ &= \delta_{h,e_x} f(\mathbf{1}_A) \mathbf{1}^x. \end{aligned} \quad (185)$$

Applying (184) and (185) to the first half of (182) gives

$$l_A \circ (\varepsilon \otimes id_A) \circ \Delta(a_g^x) = l_A \circ (\varepsilon \circ \psi^{-1} \otimes \psi^{-1}) \circ (m^{op})^* \circ \psi(a_g^x) \quad (186)$$

$$= \sum_{i,j} \alpha_{ij} v_i^*(\mathbf{1}_A) \psi^{-1}(u_j^*) \quad (187)$$

$$= \sum_{i,j} \alpha_{ij} v_i^*(v_1) \psi^{-1}(u_j^*) \quad (188)$$

$$= \sum_{i,j} \alpha_{1j} \psi^{-1}(u_j^*) \quad (189)$$

$$= \sum_{i,j} \alpha_j \psi^{-1}(u_j^*) \quad (190)$$

$$= a_g^x. \quad (191)$$

The proof of the other half of (182) is entirely similar.

Lastly, for co-commutativity, we need to show that

$$c_{A,A} \circ \Delta(a) = \Delta(a) \quad \forall a \in A. \quad (192)$$

Again, by linearity, it suffices to prove (192) for the case when  $a = a_g^x \in A_g^x$ . To start, note that by applying  $\psi^{-1}$  to both sides of part (iii) of Lemma 3.20 (and using the definition of  $\rho$ ), it follows that

$$\rho(\gamma_g^x) \circ \psi^{-1} = \psi^{-1} \circ \rho(\gamma_{x^{-1}g^{-1}x}^{x^{-1}})^*. \quad (193)$$

Next, note that if  $b_{g_1^{-1}}^x \in A_{g_1^{-1}}^x$  and  $c_{g_2^{-1}}^x \in A_{g_2^{-1}}^x$  with  $g_1 g_2 = g$ , then

$$\begin{aligned} (m^{op})^* \circ \psi(a_g^x)(b_{g_1^{-1}}^x \otimes c_{g_2^{-1}}^x) &= \psi(a_g^x)(c_{g_2^{-1}}^x \bullet b_{g_1^{-1}}^x) \\ &= \psi(a_g^x)((\varphi(g_2^{-1})b_{g_1^{-1}}^x) \bullet c_{g_2^{-1}}^x) \\ &= \psi(a_g^x)((\rho(\gamma_{g_2^{-1}g_1^{-1}g_2}^{g_2^{-1}})b_{g_1^{-1}}^x) \bullet (\rho(\gamma_{g_2^{-1}}^{e_x})c_{g_2^{-1}}^x)) \end{aligned} \quad (194)$$

where the second equality follows from axiom (vii) of Definition 3.1 and the third equality follows directly from the definition of  $\rho$ . (194) then implies that

$$(m^{op})^* \circ \psi(a_g^x) = \left[ \sum_{g_1 g_2 = g} \rho(\gamma_{g_2^{-1}}^{g_1^{-1}})^* \otimes \rho(\gamma_{g_1^{-1}}^{e_x})^* \right] \circ m^* \circ \psi(a_g^x). \quad (195)$$

Now let  $\tau : A \hat{\otimes} A \rightarrow A \hat{\otimes} A$  be the  $k$ -linear map defined by  $\tau(a^y \otimes b^y) := b^y \otimes a^y$ . The proof of (192) then follows from (193) and (195):

$$\begin{aligned}
 c_{A,A} \circ \Delta(a_g^x) &= \tau \circ \left[ \left( \sum_{y \in \mathcal{G}_0} \sum_{h,l \in \Gamma^y} \rho(\gamma_h^{e_y}) \circ \psi^{-1} \otimes \rho(\gamma_l^h) \circ \psi^{-1} \right) \circ (m^{op})^* \circ \psi(a_g^x) \right] \\
 &= \tau \circ \left[ \left( \sum_{g_1 g_2 = g} \rho(\gamma_{g_1}^{e_x}) \circ \psi^{-1} \otimes \rho(\gamma_{g_1 g_2 g_1^{-1}}^{g_1}) \circ \psi^{-1} \right) \circ (m^{op})^* \circ \psi(a_g^x) \right] \\
 &= \tau \circ \left[ \left( \sum_{g_1 g_2 = g} \psi^{-1} \circ \rho(\gamma_{g_1}^{e_x})^* \otimes \psi^{-1} \circ \rho(\gamma_{g_2^{-1}}^{g_1^{-1}})^* \right) \circ (m^{op})^* \circ \psi(a_g^x) \right] \\
 &= (\psi^{-1} \otimes \psi^{-1}) \circ \left[ \left( \sum_{g_1 g_2 = g} \rho(\gamma_{g_2^{-1}}^{g_1^{-1}})^* \otimes \rho(\gamma_{g_1}^{e_x})^* \right) \circ m^* \circ \psi(a_g^x) \right] \\
 &= (\psi^{-1} \otimes \psi^{-1}) \circ (m^{op})^* \circ \psi(a_g^x) \\
 &= \Delta(a_g^x)
 \end{aligned}$$

where the third equality follows from (193) and the fifth equality follows from (195).  $\square$

**Lemma 3.24.** *Suppose  $\langle \mathcal{G}, (A, \bullet, \mathbf{1}_A), \eta, \varphi \rangle$  is a  $\mathcal{G}$ -FA . Then*

- (i)  $\varphi(g^{-1})|_{A_g^x} = id_{A_g^x}$ ,
- (ii)  $a_g^x \bullet b_{g^{-1}}^x = b_{g^{-1}}^x \bullet a_g^x$  for all  $a_g^x \in A_g^x$ ,  $b_{g^{-1}}^x \in A_{g^{-1}}^x$ , and
- (iii)  $\eta$  is symmetric

for all  $x \in \mathcal{G}_0$ ,  $g \in \Gamma^x$ .

*Proof.* Part (i) follows immediately from axiom (viii) of Definition 3.1 and the fact that  $\varphi(e_x) = id_{A^x}$ . Part (ii) then follows from part (i) of Lemma 3.24 and axiom (vii) of Definition 3.1. For (iii), we have

$$\begin{aligned}
 \eta(a_g^x, b_h^x) &= \eta(a_g^x \bullet b_h^x, \mathbf{1}_A) \\
 &= \eta((\varphi(g)b_h^x) \bullet a_g^x, \mathbf{1}_A) \\
 &= \eta(\varphi(g)b_h^x, a_g^x) \\
 &= \eta(b_h^x, \varphi(g^{-1})a_g^x) \\
 &= \eta(b_h^x, a_g^x)
 \end{aligned}$$

where the fourth equality follows from axiom (iv) of Definition 3.1 and the last equality follows from part (i) of Lemma 3.24.  $\square$

**Lemma 3.25.** *Suppose  $\langle \mathcal{G}, (A, \bullet, \mathbf{1}_A), \eta, \varphi \rangle$  is a  $\mathcal{G}$ -FA and  $\{u_i\}$  is any basis of  $A^x$  where  $u_i \in A_{g_i}^x$  for some  $g_i \in \Gamma^x$ . Let*

$$\hat{u}_i := \psi^{-1}(u_i^*) \quad (196)$$

where  $\{u_i^*\}$  is the dual basis of  $\{u_i\}$ . Then

- (i)  $\hat{u}_i \in A_{g_i^{-1}}^x$ ;

- (ii)  $\{\widehat{u}_i\}$  is a basis of  $A^\times$ ; and
- (iii)  $\psi(u_i) = \widehat{u}_i^*$ .

*Proof.* Parts (i) and (ii) are both immediate consequences of Lemma 3.20. By part (iii) of Lemma 3.24, we also have

$$\begin{aligned}
 \psi(u_i)(\widehat{u}_j) &= \eta(u_i, \widehat{u}_j) \\
 &= \eta(\widehat{u}_j, u_i) \\
 &= \psi(\widehat{u}_j)(u_i) \\
 &= u_j^*(u_i) \\
 &= \delta_{ij},
 \end{aligned}$$

which proves (iii). □

**Proposition 3.26.** *Suppose  $\langle \mathcal{G}, (A, \bullet, 1_A), \eta, \varphi \rangle$  is a  $\mathcal{G}$ -FA and  $((\rho, A), m, \mu)$  and  $((\rho, A), \Delta, \varepsilon)$  are the algebra and coalgebra objects given respectively in Propositions 3.17 and 3.23. Then  $((\rho, A), m, \Delta, \mu, \varepsilon)$  is a Frobenius object in  $\text{Rep}(D(k[\mathcal{G}]))$ .*

*Proof.* The only thing we have left to check are the Frobenius relations:

$$\Delta \circ m = (m \otimes id_A) \circ (id_A \otimes \Delta) \quad (197)$$

$$\Delta \circ m = (id_A \otimes m) \circ (\Delta \otimes id_A) \quad (198)$$

To start, let  $\{u_i\}$  be any basis of  $A^\times$  where  $u_i \in A_{g_i}^\times$  for some  $g_i \in \Gamma^\times$  and let  $\{\widehat{u}_i\}$  be the basis given by Lemma 3.25. Then

$$u_i \bullet u_j = \sum_t C_{ij}^t u_t \quad (199)$$

$$\widehat{u}_l \bullet \widehat{u}_m = \sum_t \widehat{C}_{lm}^t \widehat{u}_t \quad (200)$$

for some  $C_{ij}^t, \widehat{C}_{lm}^t \in k$  where we note that  $C_{ij}^t = 0$  if  $g_t \neq g_i g_j$  and  $\widehat{C}_{lm}^t = 0$  if  $g_t \neq g_m g_l$ . Next, note that

$$\widehat{u}_l \bullet u_i = \sum_s C_{is}^l \widehat{u}_s. \quad (201)$$

(201) follows from the fact that if  $\alpha^s \in k$  is the scalar multiplying  $\widehat{u}_s$  then

$$\begin{aligned}
 \alpha^s &= \widehat{u}_s^*(\widehat{u}_l \bullet u_i) \\
 &= \psi(u_s)(\widehat{u}_l \bullet u_i) \\
 &= \eta(u_s, \widehat{u}_l \bullet u_i) \\
 &= \eta(\widehat{u}_l \bullet u_i, u_s) \\
 &= \eta(\widehat{u}_l, u_i \bullet u_s) \\
 &= \sum_t C_{is}^t \eta(\widehat{u}_l, u_t) \\
 &= \sum_t C_{is}^t \psi(\widehat{u}_l)(u_t) \\
 &= \sum_t C_{is}^t u_l^*(u_t) \\
 &= C_{is}^l.
 \end{aligned}$$

We now prove (197). (The proof of (198) is similar.) To do this, it suffices to show that

$$\Delta(u_i \bullet u_j) = (m \otimes id_A) \circ (id_A \otimes \Delta)(u_i \otimes u_j). \quad (202)$$

It follows from the definition of  $\Delta$  as well as that of  $\{u_i\}$  and  $\{\widehat{u}_i\}$  that

$$\Delta(a^x) = \sum_{l,m} \eta(a^x, \widehat{u}_m \bullet \widehat{u}_l) u_l \otimes u_m. \quad (203)$$

Hence, the right side of (202) is

$$\sum_{l,m} \eta(u_j, \widehat{u}_m \bullet \widehat{u}_l) (u_i \bullet u_l) \otimes u_m. \quad (204)$$

Computing the left side of (202) gives

$$\begin{aligned}
 \Delta(u_i \bullet u_j) &= \sum_{l,m} \eta(u_i \bullet u_j, \widehat{u}_m \bullet \widehat{u}_l) u_l \otimes u_m \\
 &= \sum_{l,m} \eta(\widehat{u}_m \bullet \widehat{u}_l, u_i \bullet u_j) u_l \otimes u_m \\
 &= \sum_{l,m} \eta(\widehat{u}_m \bullet (\widehat{u}_l \bullet u_i), u_j) u_l \otimes u_m \\
 &= \sum_{l,m} \sum_s C_{is}^l \eta(\widehat{u}_m \bullet \widehat{u}_s, u_j) u_l \otimes u_m \\
 &= \sum_{m,s} \eta(\widehat{u}_m \bullet \widehat{u}_s, u_j) \left( \sum_l C_{is}^l u_l \right) \otimes u_m \\
 &= \sum_{m,s} \eta(\widehat{u}_m \bullet \widehat{u}_s, u_j) (u_i \bullet u_s) \otimes u_m.
 \end{aligned}$$

$$= \sum_{m,s} \eta(u_j, \hat{u}_m \bullet \hat{u}_s) (u_i \bullet u_s) \otimes u_m.$$

Comparing the last line of the above calculation with (204) shows that the left and right sides of (202) are indeed equal.  $\square$

**Proposition 3.27.** *Suppose  $\langle \mathcal{G}, (A, \bullet, \mathbf{1}_A), \eta, \varphi \rangle$  is a  $\mathcal{G}$ -FA and  $((\rho, A), m, \Delta, \mu, \varepsilon)$  is the Frobenius object of Proposition 3.26. Then  $((\rho, A), m, \Delta, \mu, \varepsilon)$  satisfies conditions (1) and (2) of Theorem 3.4.*

*Proof.* For condition (1), let  $a = \sum_{x \in \mathcal{G}_0} \sum_{g \in \Gamma^x} a_g^x$ . Then

$$\begin{aligned} \sum_{x \in \mathcal{G}_0} \sum_{g \in \Gamma^x} \rho(\gamma_g^x) a &= \sum_{x \in \mathcal{G}_0} \sum_{g \in \Gamma^x} \varphi(g) a_g^x \\ &= \sum_{x \in \mathcal{G}_0} \sum_{g \in \Gamma^x} a_g^x \\ &= a \end{aligned}$$

where the first equality follows from the definition of  $\rho$  and the second equality follows from axiom (viii) of Definition 3.1.

For condition (2), note that

$$\mathrm{Tr} \left( l_c \circ \rho(\gamma_{hgh^{-1}}^h) \right) = \mathrm{Tr} \left( l_c \circ \varphi(h)|_{A_g^x} : A_g^x \rightarrow A_g^x \right) \quad (205)$$

and

$$\mathrm{Tr} \left( \rho(\gamma_h^{g^{-1}}) \circ l_c \circ \rho(\gamma_h^{e_x}) \right) = \mathrm{Tr} \left( \varphi(g^{-1}) \circ l_c|_{A_h^x} : A_h^x \rightarrow A_h^x \right). \quad (206)$$

Condition (2) then follows from axiom (ix) of Definition 3.1.  $\square$

**3.3. Conclusion.** The proof of Theorem 3.4 now follows from Propositions 3.14 and 3.26. Specifically, Proposition 3.14 shows that every Frobenius object in  $\mathrm{Rep}(D(k[\mathcal{G}]))$  satisfying the two conditions of Theorem 3.4 induces a  $\mathcal{G}$ -FA. This proves the second half of Theorem 3.4. In addition, **every**  $\mathcal{G}$ -FA is derived from a Frobenius object in  $\mathrm{Rep}(D(k[\mathcal{G}]))$  which satisfies conditions (1) and (2) of Theorem 3.4. To see this, let  $\mathcal{A}$  be any  $\mathcal{G}$ -FA and use Proposition 3.26 to represent  $\mathcal{A}$  as a Frobenius object in  $\mathrm{Rep}(D(k[\mathcal{G}]))$ . By Proposition 3.27, this Frobenius object satisfies conditions (1) and (2) of Theorem 3.4. It is easy to check that if Proposition 3.14 is applied to the aforementioned Frobenius object, the resulting  $\mathcal{G}$ -FA is exactly  $\mathcal{A}$  and this proves the first part of Theorem 3.4.

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